HIGH-SPEED OUTPUTS FROM A STRAPDOWN IMU FOR TWO-SPEED ATTITUDE/VELOCITY UPDATING IN A CENTRAL COMPUTER

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ABSTRACT

This article defines analytical formulas for combining a sequence of high-speed calculated coning/sculling (or rotation/translation vectors) with integrated angular-rate/specific-force-acceleration increments into the form required for lower-speed strapdown inertial navigation attitude/velocity updating. This has been the basic "two-speed" concept traditionally embedded within strapdown inertial navigation system (INS) software. This article expands the two-speed concept into two physically separate operations, the high-speed portion resident within a strapdown Inertial Measurement Unit (IMU), the lower-speed portion resident within a separate central computer receiving the IMU outputs. The focus is to analytically identify the IMU high-speed computations, outputs, and "down-summing" interface routine required in the central computer for conversion into the lower-speed equivalent for attitude/velocity updating. A universal IMU high-speed output interface format is then defined, compatible with down-summing into several attitude/velocity computation approaches, each differing by analytical approximations used in their velocity update formulation. This article is an expansion of the approach described in a previous article for down-summing IMU provided high-speed coning (or rotation) vectors and integrated-angular-rate-increments, into the form needed for lower-speed attitude updating within a central computer.

INTRODUCTION

In a modern-day strapdown inertial navigations system (INS), a two-speed architecture has been commonly used for attitude and velocity updating. With the two-speed approach, attitude/ velocity are updated at a basic computation speed using inputs generated at high-speed. The high-speed algorithms are designed to accurately measure high frequency dynamic angular-rotation and linear-acceleration effects ("coning" motion for attitude updating, "sculling" motion for velocity updating). The general concept was originated in 1966 for attitude updating as a means for reducing computer throughput [1]. An important subsequent contribution in [2] was to use an exact algorithm for attitude updating coupled with an approximate high-speed coning correction based on the Goodman/Robinson theorem [3]. More sophisticated two-speed digital attitude updating approaches have evolved since [4 - 11] based on simplified versions of the Laning rotation vector rate equation [12].

Ref. [5] presented a two-speed analytical approach for velocity updating. Refs. [9, 13, 14 (Sects. 7.2), 15] expanded the [2] exact two-speed attitude updating architecture to velocity updating coupled with associated high-speed sculling algorithms. For the unified attitude/velocity/position updating architecture presented in [16], the sculling contribution was imbedded within a "velocity-translation vector", analogous to the coning contribution for attitude updating imbedded within the "rotation vector". (Note: Refs. [11, 14 (Sects. 7.3.3 & 19.1.4), 16] also describe a "position translation vector" for high resolution position updating with an associated high-speed "scrolling" computation analogous to coning/sculling for attitude/velocity updating.) Ref. [15] showed how a fixed gain digital filter can be designed and applied to strapdown accelerometer (or gyro) outputs so that sculling algorithms can be modified to accommodate dynamic phase shift between gyros/accelerometers and between angular/linear vibrations. Ref. [17] identified the common analytical structure that exists within a class of coning versus sculling algorithms, and [18] provided a general formula for converting previously designed coning algorithms into their equivalent sculling counterparts.

For many years, high accuracy strapdown technology contained navigation parameter updating operations within the INS computer with associated results provided as outputs. After basic strapdown computational methods had reached a common high level of sophistication (i.e., "documented analytical agreement"), new INS architectures expanded to include a strapdown Inertial Measurement Unit (IMU) as an option, a separate package containing gyros, accelerometers, sensor compensation related computations, while providing high-speed coning/sculling vectors and integrated angular-rate/specific-force increments for output. With this approach, navigational attitude/velocity/position updating is executed within a separate centralized computer using IMU data for input. Interfacing algorithms are then required to convert ("down-sum") the IMU provided high-speed data into the lower-speed equivalents for navigation parameter updating (i.e., a physically separated two-speed approach). The required down-summing interface routines are identical to what has been applied within previous integrated strapdown INS two-speed computational structures.

Ref. [19] defined an IMU to central-computer rate-conversion function for the rotation vector, analytically showing how IMU high-speed coning and integrated-angular rate increment outputs can be down-summed into a rotation vector for lower-speed attitude updating. This article expands on [19], adding the equivalent for the sculling vector, showing how IMU high-speed computed sculling (and associated integrated angular-rate/specific-force increments) can be "down-summed" into a velocity-translation vector (or equivalent) for lower-speed velocity updating. Down-sum approaches are presented for four velocity update approaches, differing by approximations used in their analytical formulation. Based on the down-sum methods shown, a universal IMU high-speed output format is then defined that is compatible with down-summing for either of the attitude/velocity updating methods shown.

ROTATION AND VELOCITY-TRANSLATION VECTOR ANALYTICS

Angular attitude and velocity in a modern-day strapdown INS is updated at a prescribed rate based on the fundamental equation relating the attitude rate of change to angular rotation rate, and the velocity rate of change to gravity plus non-gravitational "specific force" acceleration:

$$\begin{aligned} \frac{d\,C_B^N}{dt} &= C_B^N \left(\underline{\omega}_{IB}^B \times \right) = C_{B_{n-1}}^N \, C_B^{B_{n-1}} \left(\underline{\omega}_{IB}^B \times \right) \\ \frac{d\underline{v}^N}{dt} &= \underline{g}^N + C_B^N \underline{a}_{SF}^B = \underline{g}^N + C_{B_{n-1}}^N \, C_B^{B_{n-1}} \, \underline{a}_{SF}^B \end{aligned} \tag{1}$$

where

N = Superscript indicating that the underlined vector components are projected on navigation coordinate frame N. For this article, N is defined to be non-rotating. Augmentation can be added to include prescribed rotations of the N frame [14 (Sects. 7.1.1.2 & 7.1.1.2-2)].

B = Superscript indicating that the vector components are as projected on strapdown rotating ("body") coordinate frame B, the frame to which the strapdown gyros and accelerometers are aligned.

n = Computation cycle index used for attitude/velocity updating in the INS computer.

B_{n-1} = Non-rotating coordinate frame parallel to the rotating B frame at computer cycle n-1.

$\underline{\omega}_{IB}^B$ = Angular rate vector of the B frame relative to non-rotating inertial space (IB subscript) that would be measured by the INS strapdown gyro triad.

C_B^N = Direction cosine matrix that transforms vectors from the B frame to the N frame.

$C_{B_{n-1}}^N$ = C_B^N matrix at computer cycle n-1.

$C_B^{B_{n-1}}$ = Direction cosine matrix that transforms vectors from the instantaneously rotating B frame to the non-rotating B_{n-1} coordinate frame.

$\underline{v}^N$ = Velocity (rate of change of position) in non-rotating N frame navigation coordinates.

$\underline{g}^N$ = Local gravity in non-rotating N frame coordinates.

$\underline{a}_{SF}^B$ = Non-gravitational "specific-force" acceleration in B frame coordinates, the acceleration sensed by the strapdown accelerometers.

Refs. [13 Rev 2 (Sect. 19.1.3), 15, 16] provide a high-accuracy two-speed incremental integral form of (1) for attitude/velocity updating:

$$C^N_{B_n} = C^N_{B_{n-1}} \left[I + f_{1_n} \left(\underline{\phi}_n \times \right) + f_{2_n} \left(\underline{\phi}_n \times \right)^2 \right]$$

$$f_{1_n} \equiv \frac{\sin \phi_n}{\phi_n} = 1 - \frac{\phi_n^2}{3!} + \frac{\phi_n^4}{5!} + \cdots \qquad f_{2_n} \equiv \frac{1 - \cos \phi_n}{\phi^2} = 1 - \frac{\phi_n^2}{2!} + \frac{\phi_n^4}{4!} + \cdots$$

(2)

$$\underline{v}^N_n = \underline{v}^N_{n-1} + \Delta \underline{v}^N_{g_n} + C^N_{B_{n-1}} \, \Delta \underline{v}^{B_{n-1}}_{SF_n}$$

$$\Delta \underline{v}^{B_{n-1}}_{SF_n} = \left[I + f_{2_n} \left(\underline{\phi}_n \times \right) + f_{3_n} \left(\underline{\phi}_n \times \right)^2 \right] \underline{\eta}_n$$

$$f_{2_n} \equiv \frac{1 - \cos \phi_n}{\phi^2} = 1 - \frac{\phi_n^2}{2!} + \frac{\phi_n^4}{4!} + \cdots \qquad f_{3_n} \equiv \frac{1}{\phi_n^2} \left(1 - \frac{\sin \phi_n}{\phi_n} \right) = \frac{1}{3!} - \frac{\phi_n^2}{5!} + \frac{\phi_n^4}{7!} + \cdots$$

where

$\underline{\phi}_n$ = Rotation vector for attitude update cycle n that measures angular rotation relative to non-rotating inertial space over the n-1 to n time interval.

ϕ_n = Magnitude of $\underline{\phi}_n$.

$\left(\underline{\phi}_n \times \right)$ = Cross-product operator form of $\underline{\phi}_n$ defined such that when formatted as a square matrix, its product with an arbitrary column-matrix formatted $\underline{V}$ vector equals the cross-product of $\underline{\phi}_n$ with that vector, i.e., $\left(\underline{\phi}_n \times \right) \underline{V} = \underline{\phi}_n \times \underline{V}$.

$\underline{v}^N_n$ = Velocity vector at computer cycle n.

$\Delta \underline{v}^N_{g_n}$ = Change in velocity caused by gravity over the n-1 to n velocity update time cycle interval.

$\Delta \underline{v}^{B_{n-1}}_{SF_n}$ = Change in velocity due to specific force (non-gravitational) acceleration over the n-1 to n velocity update time cycle interval as measured in the non-rotating B_{n-1} frame.

$\underline{\eta}_n$ = Velocity-translation vector that measures velocity change over the n-1 to n time interval.

The rate of change of the rotation and velocity-translation vectors in (2) is derived in [16] as:

$$
\begin{aligned}
\underline{\dot{\phi}} &= \underline{\omega}_{IB}^{B} + \frac{1}{2}\left(\underline{\phi}\times\underline{\omega}_{IB}^{B}\right) + \cdots \approx \underline{\omega}_{IB}^{B} + \frac{1}{2}\left(\underline{\alpha}\times\underline{\omega}_{IB}^{B}\right) \\
\underline{\dot{\eta}} &= \underline{a}_{SF}^{B} + \frac{1}{2}\left(\underline{\phi}\times\underline{a}_{SF}^{B} - \underline{\omega}_{IB}^{B}\times\underline{\eta}\right) + \cdots \approx \underline{a}_{SF}^{B} + \frac{1}{2}\left(\underline{\alpha}\times\underline{a}_{SF}^{B} - \underline{\omega}_{IB}^{B}\times\underline{\upsilon}\right) \qquad (3) \\
\underline{\alpha} &\equiv \int_{t_{n-1}}^{t} \underline{\omega}_{IB}^{B}\, d\tau \qquad \underline{\upsilon} \equiv \int_{t_{n-1}}^{t} \underline{a}_{SF}^{B}\, d\tau
\end{aligned}
$$

where

t = General time parameter.

t_{n-1} = Time t at the end of the last n cycle.

$\underline{\phi}$ = Rotation vector that measures angular rotation relative to non-rotating inertial space over general small time interval t_{n-1} to t.

$\underline{\alpha}$ = Integral of $\underline{\omega}_{IB}^{B}$ over the t_{n-1} to t time interval.

$\underline{\eta}$ = Velocity-translation vector over the small integration time interval in (3) from t_{n-1} to t.

$\underline{\upsilon}$ = Integral of $\underline{a}_{SF}^{B}$ over the t_{n-1} to t time interval.

τ = Dummy integration parameter.

The integral of (3) over a velocity update cycle provides the rotation/velocity-translation vectors in (2):

$$
\begin{aligned}
&\underline{\alpha}(t) \equiv \int_{t_{n-1}}^{t} \underline{\omega}_{IB}^{B}\, d\tau \qquad \underline{\upsilon}(t) = \int_{t_{n-1}}^{t} \underline{a}_{SF}^{B}\, d\tau \\
&\delta\underline{\phi}_n \equiv \frac{1}{2}\int_{t_{n-1}}^{t_n} \left(\underline{\alpha}(t)\times\underline{\omega}_{IB}^{B}\right) dt \qquad \underline{\alpha}_n \equiv \int_{t_{n-1}}^{t_n} \underline{\omega}_{IB}^{B}\, dt \\
&\delta\underline{\eta}_n \equiv \frac{1}{2}\int_{t_{n-1}}^{t_n} \left(\underline{\alpha}(t)\times\underline{a}_{SF}^{B} - \underline{\omega}_{IB}^{B}\times\underline{\upsilon}(t)\right) dt \qquad \underline{\upsilon}_n \equiv \int_{t_{n-1}}^{t_n} \underline{a}_{SF}^{B}\, dt \\
&\underline{\phi}_n = \underline{\alpha}_n + \delta\underline{\phi}_n \qquad \underline{\eta}_n = \underline{\upsilon}_n + \delta\underline{\eta}_n
\end{aligned}
\qquad (4)
$$

where

$\underline{\alpha}_n$ = Integrated angular rate increment over the n-1 to n time interval (i.e., from t_{n-1} to t_n).

$\delta\underline{\phi}_n$ = The “coning correction” vector used in calculating $\underline{\phi}_n$, i.e., the “correction” to integrated angular rate increment $\underline{\alpha}_n$.

$\underline{\upsilon}_n$ = Integrated specific force increment over the n-1 to n time interval (i.e., from t_{n-1} to t_n).

$\delta\underline{\eta}_n$ = The "sculling correction" vector used in calculating $\underline{\eta}_n$, i.e., the "correction" to integrated specific force increment $\underline{\upsilon}_n$.

An important characteristic of (2) with (4) is that it provides an exact (true) solution under constant *B* frame angular-rate/specific-force-acceleration, being only in error by approximations in the (4) coning/sculling correction calculations. Since coning and sculling corrections as analytically defined in (4) are very small in operation (under vibration), inaccuracies in (2) are negligible, particularly when compared with gyro/accelerometer errors.

For strapdown IMU output (see definition in the Introduction), coning/sculling vectors and integrated angular-rate/specific-force increments can be calculated over a faster *l* time cycle (i.e., shorter than attitude/velocity update cycle *n*) as an integral of (3) over the IMU computation *l* cycle time period:

$$
\begin{gathered}
\Delta\underline{\alpha}(t) \equiv \int_{t_{l-1}}^{t} \underline{\omega}_{IB}^{B}\, d\tau \qquad \Delta\underline{\upsilon}(t) \equiv \int_{t_{l-1}}^{t} \underline{a}_{SF}^{B}\, d\tau \\
\Delta\underline{\alpha}_l \equiv \Delta\underline{\alpha}(t_l) = \int_{t_{l-1}}^{t_l} \underline{\omega}_{IB}^{B}\, dt \qquad \Delta\underline{\upsilon}_l \equiv \Delta\underline{\upsilon}(t_l) = \int_{t_{l-1}}^{t_l} \underline{a}_{SF}^{B}\, dt \\
\delta\underline{\phi}_l \equiv \frac{1}{2}\int_{t_{l-1}}^{t_l} \left(\underline{\alpha}(t)\times\underline{\omega}_{IB}^{B}\right) dt \qquad \delta\underline{\eta}_l \equiv \frac{1}{2}\int_{t_{l-1}}^{t_l} \left(\Delta\underline{\alpha}(t)\times\underline{a}_{SF}^{B} - \underline{\omega}_{IB}^{B}\times\Delta\underline{\upsilon}(t)\right) dt \\
\underline{\phi}_l = \Delta\underline{\alpha}_l + \delta\underline{\phi}_l \qquad \underline{\eta}_l = \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l
\end{gathered}
\tag{5}
$$

where

l = Computer cycle index for calculating $\underline{\phi}_l$ and $\underline{\eta}_l$. There are an integer number of *l*-1 to *l* time intervals (i.e., from t_{l-1} to t_l) in each *n*-1 to *n* time interval (i.e., from t_{n-1} to t_n).

$\Delta\underline{\alpha}(t)$ = Integral of $\underline{\omega}_{IB}^{B}$ over the t_{l-1} to *t* time interval.

$\Delta\underline{\upsilon}(t)$ = Integral of $\underline{a}_{SF}^{B}$ over the t_{l-1} to *t* time interval.

$\Delta\underline{\alpha}_l$ = Integral of $\underline{\omega}_{IB}^{B}$ over the *l*-1 to *l* time interval.

$\Delta\underline{\upsilon}_l$ = Integral of $\underline{a}_{SF}^{B}$ over the *l*-1 to *l* time interval.

$\delta\underline{\phi}_l$ = The "coning" correction to integrated angular rate increment $\Delta\underline{\alpha}_l$.

$\underline{\phi}_l$ = Rotation vector for attitude update cycle *l* that measures angular rotation relative to non-rotating inertial space over the *l*-1 to *l* time interval.

$\delta\underline{\eta}_l$ = The "sculling" correction to integrated specific force increment $\Delta\underline{\upsilon}_l$.

$\underline{\eta}_l$ = Velocity-translation vector that measures velocity change over the *l*-1 to *l* time interval.

Several digital algorithms have been designed to calculate the $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ coning/sculling integrals based on assumed analytical forms of $\underline{\omega}_{IB}^{B}$ and $\underline{a}_{SF}^{B}$. For example, for $\underline{\omega}_{IB}^{B}$ and $\underline{a}_{SF}^{B}$ approximated by constant plus linear ramping functions, [13] shows that $\delta\underline{\phi}_l = \frac{1}{12}\left(\Delta\underline{\alpha}_{l-1} \times \Delta\underline{\alpha}_l\right)$ and $\delta\underline{\eta}_l = \frac{1}{12}\left(\Delta\underline{\alpha}_{l-1} \times \Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_{l-1}\right)$. More accurate $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ approximations have included additional $\Delta\underline{\alpha}_{l-i}$, $\Delta\underline{\upsilon}_{l-j}$ cross products (i.e., for *i* and *j* from 0 to greater than 1), or have been structured as a summation of weighted $\Delta\underline{\alpha}_{k-i}$, $\Delta\underline{\upsilon}_{k-j}$ cross products, each $\Delta\underline{\alpha}_{k-i}$, $\Delta\underline{\upsilon}_{k-j}$ integrated angular-rate/specific-force increment taken over a higher speed *k* cycle (Note - There are an L_{kl} number of *k* cycles in an *l* cycle, and the number *N* of sequential *k* cycles used to form $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ can be equal to or larger than L_{kl} [10].)

The next section derives algorithms for converting ("down-summing") a sequence of $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ coning/sculling correction vectors (and associated $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ integrated angular-rate/specific-force increments) generated with (5) at computation rate *l*, into single rotation/ velocity-translation vectors $\underline{\phi}_n$, $\underline{\eta}_n$ spanning the longer *n* cycle time interval required for attitude/velocity updating in (2). Additionally, the equivalent will be provided for converting a sequence of $\underline{\phi}_l$, $\underline{\eta}_l$ rotation/velocity-translation vectors into their *n* cycle equivalent $\underline{\phi}_n$, $\underline{\eta}_n$ form. The *l* cycle coning/sculling correction and rotation/velocity-translation vectors (with the integrated angular-rate/specific-force increments) in (5) represent outputs that would be provided by a strapdown IMU for central-computer down-summing into a lower *n* cycle rate for attitude/velocity updating.

Sections following the next will develop equivalent down-summing algorithms for situations where approximations to the (2) velocity equations are the basis for IMU outputs and subsequent velocity updating operations.

BASIC CONING AND SCULLING CORRECTION VECTOR DOWN-SUMMING

For the coning correction vector $\delta\underline{\phi}_n$ used in (4) for *n*-cycle attitude updating in (2), [19] derived a down-summing routine to generate $\delta\underline{\phi}_n$ from a sequence of $\delta\underline{\phi}_l$ coning correction vectors and $\Delta\underline{\alpha}_l$ integrated angular rate increments provided at the higher *l* cycle rate from a strapdown IMU:

$$\underline{\alpha}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\alpha}_l \qquad \delta\underline{\phi}_n = \sum_{t_{n-1}}^{t_n} \left[\delta\underline{\phi}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1} \times \Delta\underline{\alpha}_l\right) \right] \tag{6}$$

The $\delta\underline{\phi}_n$ result from (6) would then be used in place of $\delta\underline{\phi}_n$ in (4) to generate rotation vector $\underline{\phi}_n$ for attitude/velocity updating in (2). Ref. [19] also derives the equivalent of (6) for down-summing a sequence of $\underline{\phi}_l$ rotation vectors directly into $\underline{\phi}_n$:

$$\underline{\alpha}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\alpha}_l \qquad \underline{\phi}_n = \sum_{t_{n-1}}^{t_n} \left[\underline{\phi}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1} \times \Delta\underline{\alpha}_l \right) \right] \tag{7}$$

Note that for $l = 1$, the $\underline{\alpha}_{l-1}$ term in (6) and (7) is zero as can be seen from the following:

$$\begin{gathered} \underline{\alpha}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\alpha}_l = \left(\sum_{t_{n-1}}^{t_l} \Delta\underline{\alpha}_l \right) - \Delta\underline{\alpha}_l \\ \text{For } l = 1: \quad \underline{\alpha}_{l-1} = \left(\sum_{t_{n-1}}^{t_{l=1}} \Delta\underline{\alpha}_l \right) - \Delta\underline{\alpha}_{l=1} = \Delta\underline{\alpha}_{l=1} - \Delta\underline{\alpha}_{l=1} = 0 \end{gathered} \tag{8}$$

For the sculling correction vector $\delta\underline{\eta}_n$ used in (4) for *n*-cycle velocity updating in (2), the remainder of this section derives a down-summing routine to generate $\delta\underline{\eta}_n$ from a sequence of $\delta\underline{\eta}_l$ sculling correction vectors and $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ integrated angular-rate/specific-force increments provided at a higher *l* cycle rate from a strapdown IMU. Alternatively, [18] could be used directly to convert (6) for the coning vector into the equivalent sculling vector down-summing routine. For comparison, the same result is derived next from (4) and (5) as in [19].

For subsequent reference, first recall that

$$\begin{gathered} \underline{\alpha}(t) \equiv \int_{t_{n-1}}^{t} \underline{\omega}_{IB}^{B}\, d\tau \qquad \underline{\upsilon}(t) = \int_{t_{n-1}}^{t} \underline{a}_{SF}^{B}\, d\tau \\ \Delta\underline{\alpha}(t) \equiv \int_{t_{l-1}}^{t} \underline{\omega}_{IB}^{B}\, d\tau \qquad \Delta\underline{\alpha}_l \equiv \Delta\underline{\alpha}(t_l) \qquad \Delta\underline{\upsilon}(t) \equiv \int_{t_{l-1}}^{t} \underline{a}_{SF}^{B}\, d\tau \qquad \Delta\underline{\upsilon}_l \equiv \Delta\underline{\upsilon}(t_l) \\ \underline{\alpha}_{l-1} \equiv \underline{\alpha}(t_{l-1}) = \sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\alpha}_l \qquad \underline{\upsilon}_{l-1} \equiv \underline{\upsilon}(t_{l-1}) = \sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\upsilon}_l \\ \underline{\alpha}_n \equiv \underline{\alpha}(t_n) = \sum_{t_{n-1}}^{t_n} \Delta\underline{\alpha}_l \qquad \underline{\upsilon}_n \equiv \underline{\upsilon}(t_n) = \sum_{t_{n-1}}^{t_n} \Delta\underline{\upsilon}_l \end{gathered} \tag{9}$$

From (9) we can write

$$\begin{gathered} \underline{\alpha}(t) = \int_{t_{n-1}}^{t} \underline{\omega}_{IB}^{B}\, d\tau = \int_{t_{n-1}}^{t_{l-1}} \underline{\omega}_{IB}^{B}\, d\tau + \int_{t_{l-1}}^{t} \underline{\omega}_{IB}^{B}\, d\tau = \underline{\alpha}_{l-1} + \Delta\underline{\alpha}(t) \\ \underline{\upsilon}(t) = \int_{t_{n-1}}^{t} \underline{a}_{SF}^{B}\, d\tau = \int_{t_{n-1}}^{t_{l-1}} \underline{a}_{SF}^{B}\, d\tau + \int_{t_{l-1}}^{t} \underline{a}_{SF}^{B}\, d\tau = \underline{\upsilon}_{l-1} + \Delta\underline{\upsilon}(t) \end{gathered} \tag{10}$$

Substituting (10) into the (4) sculling correction vector $\delta\underline{\eta}_n$ expression then finds

$$\begin{aligned}\delta\underline{\eta}_n &= \frac{1}{2}\int_{t_{n-1}}^{t_n}\left(\underline{\alpha}(t)\times\underline{a}_{SF}^B - \underline{\omega}_{IB}^B\times\underline{\upsilon}(t)\right)dt = \frac{1}{2}\sum_{t_{n-1}}^{t_n}\int_{t_{l-1}}^{t_l}\left(\underline{\alpha}(t)\times\underline{a}_{SF}^B - \underline{\omega}_{IB}^B\times\underline{\upsilon}(t)\right)dt \\ &= \frac{1}{2}\sum_{t_{n-1}}^{t_n}\int_{t_{l-1}}^{t_l}\left[\left(\underline{\alpha}_{l-1}+\Delta\underline{\alpha}(t)\right)\times\underline{a}_{SF}^B - \underline{\omega}_{IB}^B\times\left(\underline{\upsilon}_{l-1}+\Delta\underline{\upsilon}(t)\right)\right]dt \\ &= \sum_{t_{n-1}}^{t_n}\frac{1}{2}\int_{t_{l-1}}^{t_l}\left(\Delta\underline{\alpha}(t)\times\underline{a}_{SF}^B - \underline{\omega}_{IB}^B\times\Delta\underline{\upsilon}(t)\right)dt + \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\underline{\alpha}_{l-1}\times\int_{t_{l-1}}^{t_l}\underline{a}_{SF}^B\,dt - \left(\int_{t_{l-1}}^{t_l}\underline{\omega}_{IB}^B\,dt\right)\times\underline{\upsilon}_{l-1}\right]\end{aligned} \tag{11}$$

Eq. (11) simplifies by substituting definitions from (9) for particular terms:

$$\underline{\alpha}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}}\Delta\underline{\alpha}_l \qquad \underline{\upsilon}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}}\Delta\underline{\upsilon}_l \qquad \delta\underline{\eta}_n = \sum_{t_{n-1}}^{t_n}\left[\delta\underline{\eta}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1}\times\Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l\times\underline{\upsilon}_{l-1}\right)\right] \tag{12}$$

The $\delta\underline{\eta}_n$ result from (12) would then be used in place of $\delta\underline{\eta}_n$ in (4) to generate velocity-translation vector $\underline{\eta}_n$ for velocity updating in (2).

The equivalent of rotation vector $\underline{\phi}_n$ down-summing formula (7) is also easily derived from (12) for velocity-translation vector $\underline{\eta}_n$ by recognizing from (4) with (9) that

$$\underline{\upsilon}_n = \sum_{t_{n-1}}^{t_n}\Delta\underline{\upsilon}_l \qquad \underline{\eta}_n = \underline{\upsilon}_n + \delta\underline{\eta}_n = \left(\sum_{t_{n-1}}^{t_n}\Delta\underline{\upsilon}_l\right) + \delta\underline{\eta}_n \tag{13}$$

Substituting $\delta\underline{\eta}_n$ from (12) with (5) for $\underline{\eta}_l$ in (13) obtains

$$\begin{aligned}\underline{\eta}_n &= \sum_{t_{n-1}}^{t_n}\Delta\underline{\upsilon}_l + \sum_{t_{n-1}}^{t_n}\left[\delta\underline{\eta}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1}\times\Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l\times\underline{\upsilon}_{l-1}\right)\right] \\ &= \sum_{t_{n-1}}^{t_n}\left[\Delta\underline{\upsilon}_l + \delta\underline{\eta}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1}\times\Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l\times\underline{\upsilon}_{l-1}\right)\right]\end{aligned} \tag{14}$$

Then from (14) with (5) for $\underline{\eta}_l$, the equivalent to (12) becomes

$$\underline{\alpha}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}}\Delta\underline{\alpha}_l \qquad \underline{\upsilon}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}}\Delta\underline{\upsilon}_l \qquad \underline{\eta}_n = \sum_{t_{n-1}}^{t_n}\left[\underline{\eta}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1}\times\Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l\times\underline{\upsilon}_{l-1}\right)\right] \tag{15}$$

Eqs. (12) and (15) are down-summing routines for calculating the *n* cycle sculling correction vector $\delta\underline{\eta}_n$ or the velocity-translation vector $\underline{\eta}_n$ used in (4) for *n*-cycle velocity updating in (2). Comparing (12) and (15) with the (6), (7) coning/rotation vector equivalents shows the similarity in form between the two expressions. Ref. [18] analytically shows how this similarity can be used to directly translate (6), (7) into (12), (15), thus bi-passing the (9) – (15) derivation process.

DOWN-SUMMING REQUIREMENTS FOR ALTERNATE VELOCITY UPDATING

This section discusses down-summing requirements when the velocity updating computation differs from the "exact" version provided in (2) with (4). Three such examples are provided next, denoted as Variation 1, 2, and 3.

VARIATION 1

For Variation 1, an approximate equivalent of (2) can be derived as in [14 Sect. 7.2.2.2.1], or equivalently, by incorporating approximations in (2). For the latter approach, the coning correction vector used when calculating rotation vector $\underline{\phi}_n$ is neglected so that $\underline{\phi}_n$ becomes approximately $\underline{\alpha}_n$, the *n* cycle integrated angular rate increment. Then with (4), $\Delta\underline{v}_{SF_n}^{B_{n-1}}$ in (2) approximates as

$$\underline{\alpha}_n = \sum_{t_{n-1}}^{t_n} \Delta\underline{\alpha}_l \qquad \underline{\upsilon}_n = \sum_{t_{n-1}}^{t_n} \Delta\underline{\upsilon}_l$$

$$\begin{aligned} \Delta\underline{v}_{SF_n}^{B_{n-1}} &\approx \left[I + f_{2_n}\left(\underline{\alpha}_n \times\right) + f_{3_n}\left(\underline{\alpha}_n \times\right)^2 \right]\left(\underline{\upsilon}_n + \delta\underline{\eta}_n\right) \\ &\approx \underline{\upsilon}_n + \left[f_{2_n}\left(\underline{\alpha}_n \times\right) + f_{3_n}\left(\underline{\alpha}_n \times\right)^2 \right]\underline{\upsilon}_n + \delta\underline{\eta}_n \end{aligned} \tag{16}$$

$$f_{2_n} \equiv \frac{1-\cos\alpha_n}{\alpha_n^2} = \frac{1}{2!} - \frac{\alpha_n^2}{4!} + \frac{\alpha_n^4}{6!} + \cdots \qquad f_{3_n} \approx \frac{1}{\alpha_n^2}\left(1 - \frac{\sin\alpha_n}{\alpha_n}\right) = \frac{1}{3!} - \frac{\alpha_n^2}{5!} + \frac{\alpha_n^4}{7!} + \cdots$$

The $\delta\underline{\eta}_n$ sculling vector in (16) would be calculated as in (4) with (12) for the corresponding down-summed *l* cycle formula.

The term in (16) between $\underline{\upsilon}_n$ and $\delta\underline{\eta}_n$ has been denoted as "exact rotation compensation" [14 Sect. 7.2.2.2.1] in that the velocity updating result would be exact under constant *B* frame angular-rate/specific-force conditions for which $\delta\underline{\eta}_n$ sculling is zero, and coning is zero (hence, $\underline{\phi}_n$ would exactly equal $\underline{\alpha}_n$). Errors in (16) are minimal compared with the (2) form because the neglected coning correction product effect is generally very small.

VARIATION 2

Variation 2 is a further approximation of (16), in which f_{2_n} is approximated as 1/2 and f_{3_n} is equated to zero:

$$\underline{\alpha}_n = \sum_{t_{n-1}}^{t_n} \Delta\underline{\alpha}_l \qquad \underline{\upsilon}_n = \sum_{t_{n-1}}^{t_n} \Delta\underline{\upsilon}_l \qquad \Delta\underline{v}_{SF_n}^{B_{n-1}} \approx \underline{\upsilon}_n + \frac{1}{2}\underline{\alpha}_n \times \underline{\upsilon}_n + \delta\underline{\eta}_n \tag{17}$$

The $\delta\underline{\eta}_n$ sculling computation in (17) would be calculated as in (4) with (12) for the corresponding down-summed *l* cycle formula.

The term in (17) between $\underline{\upsilon}_n$ and $\delta\underline{\eta}_n$ has been denoted as "rotation compensation" [14 Sect. 7.2.2.2]. The f_{2_n} and f_{3_n} approximations in (17) are its principle error source, being of concern when $\underline{\alpha}_n$ is large (e.g., under high angular rates when using a lower *n* cycle time interval for velocity updating).

VARIATION 3

As in [14 Sect. 7.2.2.2], Variation 3 derives directly from the integral of (1) by defining an equivalent $\Delta\underline{v}_{SF_n}^{B_{n-1}*}$ for $\Delta\underline{v}_{SF_n}^{B_{n-1}}$ in (2), and using a first order approximation for $C_B^{B_{n-1}}$:

$$\Delta\underline{v}_{SF_n}^{B_{n-1}*} \equiv \int_{t_{n-1}}^{t_n} C_B^{B_{n-1}}\, \underline{a}_{SF}^B\, dt \approx \underline{\upsilon}_n + \int_{t_{n-1}}^{t_n} \left(\underline{\alpha}(t)\times\right) \underline{a}_{SF}^B\, dt = \underline{\upsilon}_n + \int_{t_{n-1}}^{t_n} \underline{\alpha}(t)\, \underline{a}_{SF}^B\, dt \tag{18}$$

The integral in (18) can be calculated from the $\delta\underline{\eta}_n$ pure sculling correction term by applying the integral of the [14, Eq. (7.2.2.2-21)] identity:

$$\int_{t_{n-1}}^{t_n} \underline{\alpha}(t)\times\underline{a}_{SF}^B\, dt = \frac{1}{2}\underline{\alpha}_n \times \underline{\upsilon}_n + \frac{1}{2}\int_{t_{n-1}}^{t_n} \left(\underline{\alpha}(t)\times\underline{a}_{SF}^B - \underline{\omega}_{IB}^B \times \underline{\upsilon}(t)\right) dt = \frac{1}{2}\underline{\alpha}_n \times \underline{\upsilon}_n + \delta\underline{\eta}_n \tag{19}$$

With (19), $\Delta\underline{v}_{SF_n}^{B_{n-1}*}$ in (18) is

$$\Delta\underline{v}_{SF_n}^{B_{n-1}*} = \underline{\upsilon}_n + \frac{1}{2}\underline{\alpha}_n \times \underline{\upsilon}_n + \delta\underline{\eta}_n \tag{20}$$

and the equivalent of (2) becomes with (9):

$$\underline{v}_n^N \approx \underline{v}_{n-1}^N + \Delta\underline{v}_{g_n}^N + C_{B_{n-1}}^N \, \Delta\underline{v}_{SF_n}^{B_{n-1}*}$$

$$\underline{\alpha}_n = \sum_{t_{n-1}}^{t_n} \Delta\underline{\alpha}_l \qquad \underline{\upsilon}_n = \sum_{t_{n-1}}^{t_n} \Delta\underline{\upsilon}_l \qquad \Delta\underline{v}_{SF_n}^{B_{n-1}*} = \underline{\upsilon}_n + \frac{1}{2}\underline{\alpha}_n \times \underline{\upsilon}_n + \delta\underline{\eta}_n \tag{21}$$

Eq. (21) for $\Delta\underline{v}_{SF_n}^{B_{n-1}*}$ is identical to the Variation 2 Eq. (17) approximation for $\Delta\underline{v}_{SF_n}^{B_{n-1}}$. Thus, the Variation 3 approximation in (18) for $C_B^{B_{n-1}}$ is equivalent to the $f_{2_n} = 1/2$ and $f_{3_n} = 0$ approximations in (17) for Variation 2. Hence, Variation 3 is equivalent to Variation 2, and the basic down-summing routine shown previously for $\delta\underline{\eta}_n$ in (12) applies for (21), as it did for (17).

POTENTIAL UNIVERSAL IMU OUTPUT FORMAT

A universal IMU computation/output format can now be defined at the *l* cycle rate that is compatible with either of the basic or variation velocity updating approaches discussed previously. Since each of these approaches use the same $\delta\underline{\eta}_n$ sculling vector for velocity updating, the universal interface can be based on (5) (repeated next) with (6) and (12) or (7) and (15) for attitude/velocity down-summing operations:

$$\Delta\underline{\alpha}(t) = \int_{t_{l-1}}^{t} \underline{\omega}_{IB}^B \, d\tau \qquad \Delta\underline{\upsilon}(t) = \int_{t_{l-1}}^{t} \underline{a}_{SF}^B \, d\tau$$

$$\Delta\underline{\alpha}_l = \int_{t_{l-1}}^{t_l} \underline{\omega}_{IB}^B \, dt \qquad \Delta\underline{\upsilon}_l = \int_{t_{l-1}}^{t_l} \underline{a}_{SF}^B \, dt$$

$$\delta\underline{\phi}_l = \frac{1}{2}\int_{t_{l-1}}^{t_l} \left(\underline{\alpha}(t) \times \underline{\omega}_{IB}^B\right) dt \qquad \delta\underline{\eta}_l = \frac{1}{2}\int_{t_{l-1}}^{t_l} \left(\Delta\underline{\alpha}(t) \times \underline{a}_{SF}^B - \underline{\omega}_{IB}^B \times \Delta\underline{\upsilon}(t)\right) dt \tag{22}$$

$$\underline{\phi}_l = \Delta\underline{\alpha}_l + \delta\underline{\phi}_l \qquad \underline{\eta}_l = \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l$$

The $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$, $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$, $\underline{\phi}_l$, $\underline{\eta}_l$ outputs in (22) from a universal IMU output can be used to generate the down-summed result required for *n* cycle attitude and velocity updating:

For attitude updating in Basic Formula (2): Using (6) for $\delta\underline{\phi}_n$

For velocity updating in Basic Formula (2): Using (6) and (12) for $\delta\underline{\phi}_n$, $\delta\underline{\eta}_n$

For velocity updating in Variation 1 Formula (16): Using (6) and (12) for $\delta\underline{\phi}_n$, $\delta\underline{\eta}_n$

For velocity updating in Variation 2 Formula (17): Using (6) and (12) for $\delta\underline{\phi}_n$, $\delta\underline{\eta}_n$

For velocity updating in Variation 3 Formula (21): Using (6) and (12) for $\delta\underline{\phi}_n$, $\delta\underline{\eta}_n$

Note: The $\underline{\phi}_l$, $\underline{\eta}_l$ outputs in (22) have been included for cases when the central computer interface has been designed for the (7) and (15) type down-summing interface formulas.

OUTPUT ADAPTER FOR UNIVERSAL DOWN-SUMMING FROM AN EXISTING IMU

In some applications, the output from an existing IMU's output may differ from the (22) universal format. This section defines interface adapters for several situations where an existing IMU output may not contain all $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$, $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$, $\underline{\phi}_l$, $\underline{\eta}_l$ terms required in down-summing formulas (6), (7) and (12), (15).

THE FIRST SITUATION

The first situation addresses the case where an existing IMU output may contain the $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ terms and either the $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ or $\underline{\phi}_l$, $\underline{\eta}_l$ terms (but not both). For the $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ case, (6) and (12) would be used for down-summing in the central computer receiving the IMU data; for the $\underline{\phi}_l$, $\underline{\eta}_l$ case, (7) and (15) would be used for down-summing.

THE SECOND SITUATION

For this situation, the existing IMU outputs would contain $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ but not the $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ or $\underline{\phi}_l$, $\underline{\eta}_l$ terms in (22). Under this condition, $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ would be calculated from the current and previous $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ outputs using classical algorithms, e.g., from [13], $\delta\underline{\phi}_l = \frac{1}{12}\left(\Delta\underline{\alpha}_{l-1}\times\Delta\underline{\alpha}_l\right)$ and $\delta\underline{\eta}_l = \frac{1}{12}\left(\Delta\underline{\alpha}_{l-1}\times\Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l\times\Delta\underline{\upsilon}_{l-1}\right)$. Alternatively, algorithms with additional $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ terms could be used, e.g., [10], with the L_{kl} ratio between a higher speed *k* cycle rate (for $\Delta\underline{\alpha}$, $\Delta\underline{\upsilon}$ sampling) and the *l* cycle rate set to 1 (i.e., the *l* cycle data would be the source for past samples used in the algorithm). Having so calculated $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$, down-summing would then be performed with (6) and (12).

ANOTHER SITUATION

Another case has the IMU providing $\underline{\phi}_l$, $\underline{\eta}_l$ in (22), but not $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ required in (7), (15) for down-summing. To generate $\Delta\underline{\alpha}_l$ in (7), it is reasonable based on (5), to approximate $\Delta\underline{\alpha}_l = \underline{\phi}_l - \delta\underline{\phi}_l \approx \underline{\phi}_l$. Since the $\delta\underline{\phi}_l$ coning correction is small compared with $\Delta\underline{\alpha}_l$, its product with the small $\underline{\alpha}_{l-1}$, $\underline{\upsilon}_{l-1}$ terms in (7), (15) should be negligible. Similarly, $\Delta\underline{\upsilon}_l$ can be generated using the approximate $\Delta\underline{\upsilon}_l \approx \underline{\eta}_l$. The error in $\underline{\phi}_n$ and $\underline{\eta}_n$ induced by these approximations is derived in Appendices A and B as given by (A-5) and (B-6):

$$\widehat{\underline{\phi}}_n - \underline{\phi}_n = \frac{1}{2} \sum_{t_{n-1}}^{t_n} \left[\underline{\alpha}_{l-1} \times \delta \underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \right) \times \Delta \underline{\alpha}_l \right]$$
$$\widehat{\underline{\eta}}_n - \underline{\eta}_n = \frac{1}{2} \sum_{t_{n-1}}^{t_n} \left[\underline{\alpha}_{l-1} \times \delta \underline{\eta}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\eta}_l \right) \times \Delta \underline{\alpha}_l + \underline{\upsilon}_{l-1} \times \delta \underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \right) \times \Delta \underline{\upsilon}_l \right] \tag{23}$$

where $\widehat{(\ )}$ symbolizes the computed value for ( ) containing error. Under angular rate and specific-force acceleration represented as a constant plus a linear ramp in time, $\Delta \underline{\alpha}_l$, $\Delta \underline{\upsilon}_l$ in (23) can be represented by

$$\Delta \underline{\alpha}_l = \Delta \underline{\alpha}_c + \Delta \underline{\alpha}_{rmp}\, l \qquad \Delta \underline{\upsilon}_l = \Delta \underline{\upsilon}_c + \Delta \underline{\upsilon}_{rmp}\, l \tag{24}$$

where $\Delta \underline{\alpha}_c$, $\Delta \underline{\upsilon}_c$, $\Delta \underline{\alpha}_{rmp}$, $\Delta \underline{\upsilon}_{rmp}$ are constant, and *l* changes linearly from 0 to *L* over an *n* cycle. Then (C-4) and (C-5) in Appendix C, show that for constant $\delta \underline{\phi}_l$ and $\delta \underline{\eta}_l$,

$$\widehat{\underline{\phi}}_n - \underline{\phi}_n = \frac{1}{4} \left(\sum_{l=1}^{L} l\,(l-1) \right) \delta \underline{\phi}_l \times \Delta \underline{\alpha}_{rmp}$$
$$\widehat{\underline{\eta}}_n - \underline{\eta}_n = \frac{1}{4} \left(\sum_{l=1}^{L} l\,(l-1) \right) \left(\delta \underline{\eta}_l \times \Delta \underline{\alpha}_{rmp} + \delta \underline{\phi}_l \times \Delta \underline{\upsilon}_{rmp} \right) \tag{25}$$

Note in (25) that the $\Delta \underline{\alpha}_c$, $\Delta \underline{\upsilon}_c$ constant angular-rate/specific-force terms in (24) have no impact on the $\widehat{\underline{\phi}}_n$, $\widehat{\underline{\eta}}_n$ errors. For *L* from 1 to 10, the bracketed term in (25) is

L	1	2	3	4	5	6	7	8	9	10
$\sum_{l=1}^{L} l\,(l-1)$	0	2	8	20	40	70	112	168	240	330

To illustrate the magnitude of the (25) errors, consider an example where *L* = 10 for which from the previous table, $\sum_{l=1}^{L} l\,(l-1)$ is 330, and (25) becomes

$$\widehat{\underline{\phi}}_n - \underline{\phi}_n = \frac{330}{4} \delta \underline{\phi}_l \times \Delta \underline{\alpha}_{rmp} \qquad \widehat{\underline{\eta}}_n - \underline{\eta}_n = \frac{330}{4} \left(\delta \underline{\eta}_l \times \Delta \underline{\alpha}_{rmp} + \delta \underline{\phi}_l \times \Delta \underline{\upsilon}_{rmp} \right) \tag{26}$$

For worst case analysis, numerical values for the (26) terms can be defined under extreme conditions. For example, [14] Section 7.4 shows that under a severe 7.2 g random vibration, a 50 Hz isolated sensor assembly with 5% isolator mismatch and 1.4% center-of-mass offset (from balance) develops a sculling rate of 1.3 milli-g and a coning rate of 9.9 deg/hr. For an IMU computation/output l rate of 1 KHz, the $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ coning/sculling correction magnitudes in (25) would be $\left|\delta\underline{\phi}_l\right| = 9.9 \times \pi / (3600 \times 180 \times 1000) = 4.8$ E-8 rad and $\left|\delta\underline{\eta}_l\right| = 1.3 \times 32.2 / (1000 \times 1000) = 4.2$ E-5 fps. For an angular rate change over one second of 5 rad/sec (286 deg/sec) and a specific-force acceleration change over 1 second of 4 gs, the $\Delta\underline{\alpha}_{rmp}$, $\Delta\underline{\upsilon}_{rmp}$ magnitudes in (26) under the l rate of 1 KHz would be $\left|\Delta\underline{\alpha}_{rmp}\right| = 5 \times 1 / 1000 = 0.005$ rad/l-cycle and $\left|\Delta\underline{\upsilon}_{rmp}\right| = 4 \times 32.2 \times 1 / 1000 = 0.129$ fps/l-cycle. Based on these results and the assumed L value of 10, the magnitude of the three contributors to $\widehat{\underline{\phi}}_n - \underline{\phi}_n$ and $\widehat{\underline{\eta}}_n - \underline{\eta}_n$ in (26) approximate as

$$\text{For } \widehat{\underline{\phi}}_n - \underline{\phi}_n: \quad \frac{330}{4}\left|\delta\underline{\phi}_l \times \Delta\underline{\alpha}_{rmp}\right| \approx \frac{330}{4} \times 4.8\text{ E-8} \times 0.005 = 3.96\text{ E-8} = 0.020\ \mu\text{rad}$$

$$\text{For } \widehat{\underline{\eta}}_n - \underline{\eta}_n: \quad \frac{330}{4}\left|\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp}\right| \approx \frac{330}{4} \times 4.2\text{ E-5} \times 0.005 = 17.3\text{ E-6 fps}$$

$$\text{For } \widehat{\underline{\eta}}_n - \underline{\eta}_n: \quad \frac{330}{4}\left|\delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp}\right| \approx \frac{330}{4} \times 4.8\text{ E-8} \times 0.129 = 5.1\text{ E-7 fps}$$

For the hypothesized L value of 10 high-speed l cycles per n cycle, the l cycle rate of 1 KHz corresponds to an n cycle rate of 1000/10 = 100 Hz, i.e., 100 n cycles per second. For a maneuver at the previously hypothesized angular-rates/accelerations lasting for one second (i.e., 100 n cycles in one second), the previous n cycle attitude/velocity errors in (26) then translate into

$$\text{For } 100 \times \widehat{\underline{\phi}}_n - \underline{\phi}_n: \quad 100 \times \frac{330}{4}\left|\delta\underline{\phi}_l \times \Delta\underline{\alpha}_{rmp}\right| \approx 2.0\ \mu\text{rad}$$

$$\text{For } 100 \times \widehat{\underline{\eta}}_n - \underline{\eta}_n: \quad 100 \times \frac{330}{4}\left|\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp}\right| \approx 0.00173\text{ fps}$$

$$\text{For } 100 \times \widehat{\underline{\eta}}_n - \underline{\eta}_n: \quad 100 \times \frac{330}{4}\left|\delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp}\right| \approx 0.000051\text{ fps}$$

These errors are negligible compared with 40 μrad and 2 fps attitude/velocity errors experienced in a typical 1 nmph (nautical-mile-per-hour) accuracy INS.

A MORE COMPLICATED SITUATION

A more complicated situation might have the IMU <u>not providing</u> $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$, <u>providing</u> $\underline{\phi}_l$, but instead of $\underline{\eta}_l$, <u>providing</u> $\Delta\underline{v}_{SF_l}^{B_{n-1}}$, the l rate equivalent of $\Delta\underline{v}_{SF_n}^{B_{n-1}}$ (see (17) for example). As in the previous paragraph, $\Delta\underline{\alpha}_l$ can be approximated as $\Delta\underline{\alpha}_l \approx \underline{\phi}_l$, but $\Delta\underline{\upsilon}_l$ cannot be analogously approximated as $\Delta \underline{v}_{SF\,l}^{B_{n-1}}$ because of the presence of $\frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l$ rotation-compensation, i.e., from the *l* rate equivalent of (17), $\Delta \underline{v}_{SF\,l}^{B_{n-1}} \approx \Delta\underline{\upsilon}_l + \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l = \Delta\underline{\eta}_l + \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l$. To extract $\Delta\underline{\eta}_l$ from $\Delta \underline{v}_{SF\,l}^{B_{n-1}}$, an iteration procedure can be used based on a revised version of the previous formula: $\Delta\underline{\eta}_l = \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l = \Delta \underline{v}_{SF\,l}^{B_{n-1}} - \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l$. Using the $\underline{\phi}_l$ IMU input to approximate $\Delta\underline{\alpha}_l$ and the estimated $\Delta\underline{\eta}_l$ to approximate $\Delta\underline{\upsilon}_l$, the iteration formula would be

$$\widehat{\Delta\underline{\eta}}_{l\,j} = \Delta \underline{v}_{SF\,l}^{B_{n-1}} - \frac{1}{2}\underline{\phi}_l \times \widehat{\Delta\underline{\eta}}_{l\,j-1} \qquad (27)$$

where $\widehat{(\;)}$ signifies the estimated value of ( ) and *j* is the iteration cycle number. Note that the iteration process is performed within an *l* cycle so that $\Delta \underline{v}_{SF\,l}^{B_{n-1}}$ and $\underline{\phi}_l$ in (27) would be constant.

Starting with an initial value of $\hat{\underline{\eta}}_{l_0} \approx \Delta \underline{v}_{SF\,l}^{B_{n-1}}$, (D-12) of Appendix D shows that $\left(\hat{\underline{\eta}}_{l\,j} - \underline{\eta}_l\right)$, the error in (27), is given by

$$\left(\hat{\underline{\eta}}_{l\,j} - \underline{\eta}_l\right) = -\left[-\frac{1}{2}\left(\Delta\underline{\alpha}_l \times\right)\right]^{j+1} \Delta\underline{\upsilon}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \qquad (28)$$

The first term in (28) converges to a negligible value after only a few *j* iterations of (27). For example, for an *l* rate of 1 KHz under 5 rad/sec angular-rate and 4 g's specific-force, $\left|\Delta\underline{\alpha}_l\right|$ = 5 / 1000 = 0.005 rad and $\left|\Delta\underline{\upsilon}_l\right|$ = 4 × 32.2 / 1000 = 0.129 fps. Then after 4 iteration *j* cycles of (27), (28) shows that $\left(\frac{1}{2}\left|\Delta\underline{\alpha}_l\right|\right)^5 \left|\Delta\underline{\upsilon}_l\right| = \left(\frac{0.005}{2}\right)^5$ × 0.129 = 1.3 E-14 fps. After a one second maneuver, the cumulative velocity error would be 1000 × 1.3 E-14 = 1.3 E-11 fps - Forget about it !! Based on this finding, it can be safely assumed that the (27) iteration result will be $\widehat{\underline{\eta}}_l = \underline{\eta}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right)$. Using this value for $\widehat{\underline{\eta}}_n$ and with $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ approximated as $\widehat{\Delta\underline{\alpha}}_l \approx \underline{\phi}_l$, $\widehat{\Delta\underline{\upsilon}}_l \approx \widehat{\underline{\eta}}_l = \underline{\eta}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right)$, (E-6) in Appendix E shows that

$$
\begin{aligned}
\widehat{\underline{\eta}}_n - \underline{\eta}_n &= \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \\
&+ \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\underline{\alpha}_{l-1} \times \delta\underline{\eta}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta\underline{\eta}_l\right) \times \Delta\underline{\alpha}_l + \underline{\upsilon}_{l-1} \times \delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta\underline{\phi}_l\right) \times \Delta\underline{\upsilon}_l\right]
\end{aligned}
\tag{29}
$$

Using (24) for $\Delta\underline{\alpha}_l$ and $\Delta\underline{\upsilon}_l$, (F-4) in Appendix F shows that (29) leads to

$$
\begin{aligned}
\widehat{\underline{\eta}}_n - \underline{\eta}_n &= \frac{L}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_c - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_c\right) + \frac{L(1+L)}{4}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp} - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp}\right) \\
&= \frac{1}{4}\left(\sum_{l=1}^{L} l(l-1)\right)\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp} + \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp}\right)
\end{aligned}
\tag{30}
$$

The reader can numerically evaluate the (30) error using the same method performed in the previous section. As in the previous analysis, the result for (30) will also show that the $\widehat{\underline{\eta}}_n - \underline{\eta}_n$ error is negligible.

CONCLUSION

A strapdown inertial navigation two-speed attitude/velocity updating architecture can be readily separated into two physical packaging arrangements: 1. A strapdown IMU that computes/outputs high-speed coning/sculling and integrated angular-rate/specific-force vectors, and 2. A centralized computer for executing lower-speed attitude/velocity updates using down-summed versions of the IMU outputs. The form of the down-summing routines depends on approximations used in the composite velocity updating operations. A universal IMU output format is readily defined that is compatible with any of the described velocity updating operations.

APPENDIX A - DERIVING $\widehat{\underline{\phi}}_n - \underline{\phi}_n$ IN EQUATION (23)

This appendix derives the error induced in down-summing algorithm (7) when $\Delta\underline{\alpha}_l$ in is approximated by $\underline{\phi}_l$. From (7), the nominal down-summing formula is

$$
\underline{\phi}_n = \sum_{t_{n-1}}^{t_n}\left[\underline{\phi}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1} \times \Delta\underline{\alpha}_l\right)\right] = \sum_{t_{n-1}}^{t_n}\left[\underline{\phi}_l + \frac{1}{2}\left(\sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\alpha}_l\right) \times \Delta\underline{\alpha}_l\right]
\tag{A-1}
$$

With $\Delta\underline{\alpha}_l$ approximated by $\underline{\phi}_l$, the erroneously computed value for $\underline{\phi}_n$ would be

$$\widehat{\underline{\phi}}_n = \sum_{t_{n-1}}^{t_n}\left[\underline{\phi}_l + \frac{1}{2}\left(\sum_{t_{n-1}}^{t_{l-1}}\underline{\phi}_l\right)\times\underline{\phi}_l\right] \tag{A-2}$$

where $\widehat{(\;)}$ signifies the error in ( ). Substituting for $\underline{\phi}_l$ from (5) and expanding finds

$$\begin{aligned}
\widehat{\underline{\phi}}_n &= \sum_{t_{n-1}}^{t_n}\left\{\underline{\phi}_l + \frac{1}{2}\left[\sum_{t_{n-1}}^{t_{l-1}}\left(\Delta\underline{\alpha}_l + \delta\underline{\phi}_l\right)\right]\times\left(\Delta\underline{\alpha}_l + \delta\underline{\phi}_l\right)\right\} \\
&\approx \sum_{t_{n-1}}^{t_n}\left\{\underline{\phi}_l + \frac{1}{2}\left[\left(\sum_{t_{n-1}}^{t_{l-1}}\Delta\underline{\alpha}_l\right)\times\Delta\underline{\alpha}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\Delta\underline{\alpha}_l\right)\times\delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\phi}_l\right)\times\Delta\underline{\alpha}_l\right]\right\} \\
&= \sum_{t_{n-1}}^{t_n}\left(\underline{\phi}_l + \frac{1}{2}\underline{\alpha}_{l-1}\times\Delta\underline{\alpha}_l\right) + \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\underline{\alpha}_{l-1}\times\delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\phi}_l\right)\times\Delta\underline{\alpha}_l\right]
\end{aligned} \tag{A-3}$$

or with (A-1) for the (A-3) leading term,

$$\widehat{\underline{\phi}}_n = \underline{\phi}_n + \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\underline{\alpha}_{l-1}\times\delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\phi}_l\right)\times\Delta\underline{\alpha}_l\right] \tag{A-4}$$

Thus, the $\widehat{\underline{\phi}}_n$ error $\left(\widehat{\underline{\phi}}_n - \underline{\phi}_n\right)$ in (A-2), is

$$\left(\widehat{\underline{\phi}}_n - \underline{\phi}_n\right) = \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\underline{\alpha}_{l-1}\times\delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\phi}_l\right)\times\Delta\underline{\alpha}_l\right] \tag{A-5}$$

APPENDIX B - DERIVING $\widehat{\underline{\eta}}_n - \underline{\eta}_n$ IN EQUATION (23)

This appendix derives the error induced in down-summing algorithm (15) when $\Delta\underline{\alpha}_l$ and $\Delta\underline{\upsilon}_l$ are approximated by $\underline{\phi}_l$ and $\underline{\eta}_l$. From (15), the nominal down-summing formula is

$$\underline{\eta}_n = \sum_{t_{n-1}}^{t_n}\left[\underline{\eta}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1}\times\Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l\times\underline{\upsilon}_{l-1}\right)\right] \tag{B-1}$$

With the previously designated approximations, the estimated value for $\underline{\eta}_n$ in (B-1) is

$$\widehat{\underline{\eta}_n} = \sum_{t_{n-1}}^{t_n} \left[\underline{\eta}_l + \frac{1}{2} \left(\widehat{\underline{\alpha}_{l-1}} \times \widehat{\Delta \underline{\upsilon}_l} - \widehat{\Delta \underline{\alpha}_l} \times \widehat{\underline{\upsilon}_{l-1}} \right) \right] \tag{B-2}$$

with

$$\begin{aligned}
&\widehat{\Delta \underline{\alpha}_l} = \underline{\phi}_l = \Delta \underline{\alpha}_l + \delta \underline{\phi}_l \qquad \widehat{\Delta \underline{\upsilon}_l} = \underline{\eta}_l = \Delta \underline{\upsilon}_l + \delta \underline{\eta}_l \\
&\widehat{\underline{\alpha}_{l-1}} = \sum_{t_{n-1}}^{t_{l-1}} \widehat{\Delta \underline{\alpha}_l} = \sum_{t_{n-1}}^{t_{l-1}} \left(\Delta \underline{\alpha}_l + \delta \underline{\phi}_l \right) = \underline{\alpha}_{l-1} + \sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \\
&\widehat{\underline{\upsilon}_{l-1}} = \sum_{t_{n-1}}^{t_{l-1}} \widehat{\Delta \underline{\upsilon}_l} = \sum_{t_{n-1}}^{t_{l-1}} \left(\Delta \underline{\upsilon}_l + \delta \underline{\eta}_l \right) = \underline{\upsilon}_{l-1} + \sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\eta}_l
\end{aligned} \tag{B-3}$$

Substituting from (B-3), the $\left(\widehat{\underline{\alpha}_{l-1}} \times \widehat{\Delta \underline{\upsilon}_l} - \widehat{\Delta \underline{\alpha}_l} \times \widehat{\underline{\upsilon}_{l-1}} \right)$ term in (B-2) is

$$\begin{aligned}
&\widehat{\underline{\alpha}_{l-1}} \times \widehat{\Delta \underline{\upsilon}_l} - \widehat{\Delta \underline{\alpha}_l} \times \widehat{\underline{\upsilon}_{l-1}} \\
&= \left(\underline{\alpha}_{l-1} + \sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \right) \times \left(\Delta \underline{\upsilon}_l + \delta \underline{\eta}_l \right) - \left(\Delta \underline{\alpha}_l + \delta \underline{\phi}_l \right) \times \left(\underline{\upsilon}_{l-1} + \sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\eta}_l \right) \\
&\approx \underline{\alpha}_{l-1} \times \Delta \underline{\upsilon}_l + \underline{\alpha}_{l-1} \times \delta \underline{\eta}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \right) \times \Delta \underline{\upsilon}_l \\
&\quad - \Delta \underline{\alpha}_l \times \underline{\upsilon}_{l-1} - \Delta \underline{\alpha}_l \times \sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\eta}_l - \delta \underline{\phi}_l \times \underline{\upsilon}_{l-1} \\
&= \underline{\alpha}_{l-1} \times \Delta \underline{\upsilon}_l - \Delta \underline{\alpha}_l \times \underline{\upsilon}_{l-1} + \underline{\alpha}_{l-1} \times \delta \underline{\eta}_l - \delta \underline{\phi}_l \times \underline{\upsilon}_{l-1} \\
&\quad + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \right) \times \Delta \underline{\upsilon}_l - \Delta \underline{\alpha}_l \times \sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\eta}_l
\end{aligned} \tag{B-4}$$

With (B-4), (B-2) becomes

$$\widehat{\underline{\eta}}_n = \sum_{t_{n-1}}^{t_n}\left[\underline{\eta}_l + \frac{1}{2}\left(\widehat{\underline{\alpha}_{l-1}}\times\widehat{\Delta\underline{\upsilon}_l} - \widehat{\Delta\underline{\alpha}_l}\times\widehat{\underline{\upsilon}_{l-1}}\right)\right]$$

$$= \sum_{t_{n-1}}^{t_n}\left\{\underline{\eta}_l + \frac{1}{2}\left[\begin{array}{c}\underline{\alpha}_{l-1}\times\Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l\times\underline{\upsilon}_{l-1} + \underline{\alpha}_{l-1}\times\delta\underline{\eta}_l - \delta\underline{\phi}_l\times\underline{\upsilon}_{l-1} \\ +\left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\phi}_l\right)\times\Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l\times\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\eta}_l\end{array}\right]\right\}$$

$$= \sum_{t_{n-1}}^{t_n}\left[\underline{\eta}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1}\times\Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l\times\underline{\upsilon}_{l-1}\right)\right] \qquad \text{(B-5)}$$

$$+\frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\underline{\alpha}_{l-1}\times\delta\underline{\eta}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\eta}_l\right)\times\Delta\underline{\alpha}_l + \underline{\upsilon}_{l-1}\times\delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\phi}_l\right)\times\Delta\underline{\upsilon}_l\right]$$

$$= \underline{\eta}_n + \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\underline{\alpha}_{l-1}\times\delta\underline{\eta}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\eta}_l\right)\times\Delta\underline{\alpha}_l + \underline{\upsilon}_{l-1}\times\delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\phi}_l\right)\times\Delta\underline{\upsilon}_l\right]$$

showing that

$$\widehat{\underline{\eta}}_n - \underline{\eta}_n = \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\underline{\alpha}_{l-1}\times\delta\underline{\eta}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\eta}_l\right)\times\Delta\underline{\alpha}_l + \underline{\upsilon}_{l-1}\times\delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}}\delta\underline{\phi}_l\right)\times\Delta\underline{\upsilon}_l\right] \qquad \text{(B-6)}$$

APPENDIX C –EQUATION (23) ERRORS UNDER SPECIFIED ANGULAR/LINEAR MOTION

This appendix derives analytical values for the $\left(\widehat{\underline{\phi}}_n - \underline{\phi}_n\right)$ and $\left(\widehat{\underline{\eta}}_n - \underline{\eta}_n\right)$ errors in (23) when the $\delta\underline{\phi}_l$, $\delta\underline{\eta}_l$ coning/sculling corrections are constant, and the $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ integrated angular-rate/specific-force increments are characterized by a constant plus a linear ramping increase in time. If the *l* values are analytically defined to be zero at t_{n-1}, the $\Delta\underline{\alpha}_l$, $\Delta\underline{\upsilon}_l$ increments can be analytically defined as

$$\Delta\underline{\alpha}_l = \Delta\underline{\alpha}_c + \Delta\underline{\alpha}_{rmp}\, l \qquad \Delta\underline{\upsilon}_l = \Delta\underline{\upsilon}_c + \Delta\underline{\upsilon}_{rmp}\, l \qquad \text{(C-1)}$$

where $\Delta\underline{\alpha}_c$, $\Delta\underline{\alpha}_{rmp}$, $\Delta\underline{\upsilon}_c$, $\Delta\underline{\upsilon}_{rmp}$ are constants. Using (C-1), the (15) definitions for $\underline{\alpha}_{l-1}$, $\underline{\upsilon}_{l-1}$ translate into

$$\underline{\alpha}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\alpha}_l = \sum_{m=1}^{m=l-1} \Delta\underline{\alpha}_m = \sum_{m=1}^{m=l-1} \Delta\underline{\alpha}_c + \sum_{m=1}^{m=l-1} \Delta\underline{\alpha}_{rmp}\, m = \Delta\underline{\alpha}_c (l-1) + \Delta\underline{\alpha}_{rmp} \sum_{m=1}^{m=l-1} m$$
$$\underline{\upsilon}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\upsilon}_l = \sum_{m=1}^{m=l-1} \Delta\underline{\upsilon}_m = \Delta\underline{\upsilon}_c (l-1) + \Delta\underline{\upsilon}_{rmp} \sum_{m=1}^{m=l-1} m \tag{C-2}$$

where m is a dummy integer parameter for l. In (C-2), the m sequence represents an arithmetic progression with a first term a of 1, a number of terms $nmbr$ of $l-1$, and a last term lst of $l-1$. Ref. [20, Sect. I - ALGEBRA, Subsect. 10] shows that the sum $\sum_{m=1}^{m=l-1} m$ would then be $\frac{nmbr}{2}(a + lst) = \frac{(l-1)}{2}(1 + l - 1) = \frac{(l-1)\,l}{2}$. Substitution in (C-2) finds

$$\underline{\alpha}_{l-1} = \Delta\underline{\alpha}_c (l-1) + \Delta\underline{\alpha}_{rmp}\, l(l-1)/2 \qquad \underline{\upsilon}_{l-1} = \Delta\underline{\upsilon}_c (l-1) + \Delta\underline{\upsilon}_{rmp}\, l(l-1)/2 \tag{C-3}$$

Applying (C-3) in (23) then obtains for $\left(\widehat{\underline{\phi}_n} - \underline{\phi}_n\right)$:

$$\begin{aligned}
\left(\widehat{\underline{\phi}_n} - \underline{\phi}_n\right) &= \frac{1}{2} \sum_{t_{n-1}}^{t_n} \left[\underline{\alpha}_{l-1} \times \delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta\underline{\phi}_l \right) \times \Delta\underline{\alpha}_l \right] \\
&= \frac{1}{2} \sum_{t_{n-1}}^{t_n} \left\{ \begin{array}{c} \left[\Delta\underline{\alpha}_c (l-1) + \Delta\underline{\alpha}_{rmp}\, l(l-1)/2 \right] \times \delta\underline{\phi}_l \\ + (l-1)\, \delta\underline{\phi}_l \times \left(\Delta\underline{\alpha}_c + \Delta\underline{\alpha}_{rmp}\, l \right) \end{array} \right\} \\
&= \frac{1}{2} \sum_{t_{n-1}}^{t_n} \left\{ \left[l(l-1)/2 \right] - l(l-1) \right\} \Delta\underline{\alpha}_{rmp} \times \delta\underline{\phi}_l \\
&= -\frac{1}{4} \sum_{t_{n-1}}^{t_n} l(l-1) \Delta\underline{\alpha}_{rmp} \times \delta\underline{\phi}_l = \frac{1}{4} \left(\sum_{l=1}^{L} l(l-1) \right) \delta\underline{\phi}_l \times \Delta\underline{\alpha}_{rmp}
\end{aligned} \tag{C-4}$$

where L is the number of l cycles in an n cycle. Similarly for $\left(\widehat{\underline{\eta}_n} - \underline{\eta}_n\right)$, by comparison with (C-4):

$$\begin{aligned}
\widehat{\underline{\eta}_n} - \underline{\eta}_n &= \frac{1}{2} \sum_{t_{n-1}}^{t_n} \left[\underline{\alpha}_{l-1} \times \delta\underline{\eta}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta\underline{\eta}_l \right) \times \Delta\underline{\alpha}_l + \underline{\upsilon}_{l-1} \times \delta\underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta\underline{\phi}_l \right) \times \Delta\underline{\upsilon}_l \right] \\
&= \frac{1}{4} \left(\sum_{l=1}^{L} l(l-1) \right) \left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp} + \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp} \right)
\end{aligned} \tag{C-5}$$

APPENDIX D - DERIVATION OF ITERATION PROCESS ERROR EQUATION

The error in the (27) iteration process is $\underline{\hat{\eta}}_{l\,j} - \underline{\eta}_l$. An analytical solution for the error is derived by first substituting $\Delta\underline{v}_{SF\,l}^{B_{n-1}} \approx \underline{\upsilon}_l + \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l$ and $\underline{\phi}_l = \Delta\underline{\alpha}_l + \delta\underline{\phi}_l$ into (27) and expanding:

$$
\begin{aligned}
\left(\underline{\hat{\eta}}_{l\,j} - \underline{\eta}_l\right) &= \Delta\underline{v}_{SF\,l}^{B_{n-1}} - \frac{1}{2}\underline{\phi}_l \times \underline{\hat{\eta}}_{l\,j-1} - \underline{\eta}_l \\
&= \Delta\underline{v}_{SF\,l}^{B_{n-1}} - \underline{\eta}_l - \frac{1}{2}\underline{\phi}_l \times \left(\underline{\hat{\eta}}_{l\,j-1} - \underline{\eta}_l + \underline{\eta}_l\right) \\
&= \underline{\eta}_l + \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l - \underline{\eta}_l - \frac{1}{2}\underline{\phi}_l \times \left(\underline{\hat{\eta}}_{l\,j-1} - \underline{\eta}_l\right) - \frac{1}{2}\underline{\phi}_l \times \underline{\eta}_l \\
&= \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l - \frac{1}{2}\underline{\phi}_l \times \left(\underline{\hat{\eta}}_{l\,j-1} - \underline{\eta}_l\right) - \frac{1}{2}\left(\Delta\underline{\alpha}_l + \delta\underline{\phi}_l\right) \times \left(\Delta\underline{\upsilon}_l + \delta\underline{\eta}_l\right) \\
&= \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l - \frac{1}{2}\underline{\phi}_l \times \left(\underline{\hat{\eta}}_{l\,j-1} - \underline{\eta}_l\right) - \frac{1}{2}\left(\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l + \Delta\underline{\alpha}_l \times \delta\underline{\eta}_l + \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l + \delta\underline{\phi}_l \times \delta\underline{\eta}_l\right) \\
&\approx -\frac{1}{2}\underline{\phi}_l \times \left(\underline{\hat{\eta}}_{l\,j-1} - \underline{\eta}_l\right) + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right)
\end{aligned}
\tag{D-1}
$$

Thus,

$$
\left(\underline{\hat{\eta}}_{l\,j} - \underline{\eta}_l\right) + \frac{1}{2}\underline{\phi}_l \times \left(\underline{\hat{\eta}}_{l\,j-1} - \underline{\eta}_l\right) \approx \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \tag{D-2}
$$

Eq. (D-2) is a constant coefficient difference equation for estimation error $\left(\underline{\hat{\eta}}_{l\,j} - \underline{\eta}_l\right)$ that can be solved in traditional fashion as the sum of "homogeneous" plus "particular" parts.

The homogeneous solution $\left(\underline{\hat{\eta}}_{hl\,j} - \underline{\eta}_l\right)$ for (D-2) sets

$$
\left(\underline{\hat{\eta}}_{hl\,j} - \underline{\eta}_l\right) + \frac{1}{2}\underline{\phi}_l \times \left(\underline{\hat{\eta}}_{hl\,j-1} - \underline{\eta}_l\right) = 0 \tag{D-3}
$$

The solution assumed for (D-3) is of the form $\left(\underline{\hat{\eta}}_{hl\,j} - \underline{\eta}_l\right) = \Gamma^j\,\underline{A}$ where Γ is a constant matrix and $\underline{A}$ is an arbitrary constant vector. Substitution in (D-3) gives

$$\Gamma^{j}\,\underline{A}+\frac{1}{2}\left(\underline{\phi}_{l}\times\right)\Gamma^{j-1}\,\underline{A}=\Gamma\,\Gamma^{j-1}\,\underline{A}+\frac{1}{2}\left(\underline{\phi}_{l}\times\right)\Gamma^{j-1}\,\underline{A}=\left[\Gamma+\frac{1}{2}\left(\underline{\phi}_{l}\times\right)\right]\Gamma^{j-1}\,\underline{A}=0 \quad \text{(D-4)}$$

To satisfy (D-4) for arbitrary $\Gamma^{j-1}\,\underline{A}$, the bracketed term must be zero, thus $\Gamma=-\frac{1}{2}\left(\underline{\phi}_{l}\times\right)$ and

$$\left(\hat{\underline{\eta}}_{hl\,j}-\underline{\eta}_{l}\right)=\Gamma^{j}\,\underline{A}=\left[-\frac{1}{2}\left(\underline{\phi}_{l}\times\right)\right]^{j}\underline{A} \qquad \text{(D-5)}$$

The particular solution $\left(\hat{\underline{\eta}}_{pl\,j}-\underline{\eta}_{l}\right)$ for (D-2) has $\left(\hat{\underline{\eta}}_{l\,j}-\underline{\eta}_{l}\right)$ constant so that $\left(\hat{\underline{\eta}}_{l\,j-1}-\underline{\eta}_{l}\right)=\left(\hat{\underline{\eta}}_{l\,j}-\underline{\eta}_{l}\right)$. Then (D-2) becomes

$$\left(\hat{\underline{\eta}}_{pl\,j}-\underline{\eta}_{l}\right)+\frac{1}{2}\underline{\phi}_{l}\times\left(\hat{\underline{\eta}}_{pl\,j}-\underline{\eta}_{l}\right)=\left[I+\frac{1}{2}\left(\underline{\phi}_{l}\times\right)\right]\left(\hat{\underline{\eta}}_{pl\,j}-\underline{\eta}_{l}\right)=\frac{1}{2}\left(\delta\underline{\eta}_{l}\times\Delta\underline{\alpha}_{l}-\delta\underline{\phi}_{l}\times\Delta\underline{\upsilon}_{l}\right) \qquad \text{(D-6)}$$

hence,

$$\left(\hat{\underline{\eta}}_{pl\,j}-\underline{\eta}_{l}\right)=\frac{1}{2}\left[I+\frac{1}{2}\left(\underline{\phi}_{l}\times\right)\right]^{-1}\left(\delta\underline{\eta}_{l}\times\Delta\underline{\alpha}_{l}-\delta\underline{\phi}_{l}\times\Delta\underline{\upsilon}_{l}\right) \qquad \text{(D-7)}$$

The total solution to (D-2) is the sum of the homogeneous (D-5) and particular (D-7) parts:

$$\begin{aligned}\left(\hat{\underline{\eta}}_{l\,j}-\underline{\eta}_{l}\right)&=\left(\hat{\underline{\eta}}_{hl\,j}-\underline{\eta}_{l}\right)+\left(\hat{\underline{\eta}}_{pl\,j}-\underline{\eta}_{l}\right)\\&=\left[-\frac{1}{2}\left(\underline{\phi}_{l}\times\right)\right]^{j}\underline{A}+\frac{1}{2}\left[I+\frac{1}{2}\left(\underline{\phi}_{l}\times\right)\right]^{-1}\left(\delta\underline{\eta}_{l}\times\Delta\underline{\alpha}_{l}-\delta\underline{\phi}_{l}\times\Delta\underline{\upsilon}_{l}\right)\end{aligned} \qquad \text{(D-8)}$$

It remains to find the value of $\underline{A}$ in (D-8). This is achieved by setting (D-8) to the value of $\left(\hat{\underline{\eta}}_{l\,j}-\underline{\eta}_{l}\right)$ at j = 1, i.e., that matches the (D-1) value at j = 1 for the iteration process. The j = 1 value for $\left(\hat{\underline{\eta}}_{l\,j-1}-\underline{\eta}_{l}\right)$ in (D-1) is $\left(\hat{\underline{\eta}}_{l_0}-\underline{\eta}_{l}\right)$, the initial error value in the iteration process. To start the process we approximate $\hat{\underline{\eta}}_{l_0}\approx\Delta\underline{v}_{SF\,l}^{B_{n-1}}$ in (27) so that at j = 1,

$\left(\hat{\underline{\eta}}_{l_1} - \underline{\eta}_l\right) = \Delta \underline{v}_{SF\,l}^{B_{n-1}} - \frac{1}{2}\underline{\phi}_l \times \Delta \underline{v}_{SF\,l}^{B_{n-1}} - \underline{\eta}_l$. Then with $\underline{\eta}_l$ from (5) and the *l* rate equivalent of (17), $\Delta \underline{v}_{SF\,l}^{B_{n-1}} \approx \Delta \underline{\upsilon}_l + \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l$:

$$\begin{aligned}
\hat{\underline{\eta}}_{l_1} - \underline{\eta}_l &= \Delta \underline{v}_{SF\,l}^{B_{n-1}} - \frac{1}{2}\underline{\phi}_l \times \Delta \underline{v}_{SF\,l}^{B_{n-1}} - \underline{\eta}_l \\
&= \Delta\underline{\upsilon}_l + \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l - \underline{\eta}_l - \frac{1}{2}\left(\Delta\underline{\alpha}_l + \delta\underline{\phi}_l\right) \times \left(\Delta\underline{\upsilon}_l + \frac{1}{2}\Delta\underline{\alpha}_l \times \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l\right) \\
&\approx -\frac{1}{4}\left(\Delta\underline{\alpha}_l \times\right)\left(\Delta\underline{\alpha}_l \times\right)\Delta\underline{\upsilon}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \\
&= -\frac{1}{4}\left(\Delta\underline{\alpha}_l \times\right)^2 \Delta\underline{\upsilon}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right)
\end{aligned} \tag{D-9}$$

Equating (D-9) to (D-8) at *j* = 1 finds

$$\begin{aligned}
&-\frac{1}{4}\left(\Delta\underline{\alpha}_l \times\right)^2 \Delta\underline{\upsilon}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \\
&= -\frac{1}{2}\left(\underline{\phi} \times\right)\underline{A} - \frac{1}{2}\left[I + \frac{1}{2}\left(\underline{\phi}_l \times\right)\right]^{-1}\left(\Delta\underline{\alpha}_l \times \delta\underline{\eta}_l - \Delta\underline{\upsilon}_l \times \delta\underline{\phi}_l\right) \\
&\approx -\frac{1}{2}\left(\Delta\underline{\alpha}_l \times\right)\underline{A} + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right)
\end{aligned} \tag{D-10}$$

or

$$-\frac{1}{2}\left(\Delta\underline{\alpha}_l \times\right)\underline{A} = -\frac{1}{4}\left(\Delta\underline{\alpha}_l \times\right)^2 \Delta\underline{\upsilon}_l \tag{D-11}$$

For arbitrary $\Delta\underline{\alpha}_l$, (D-11) shows that $\underline{A} = \frac{1}{2}\left(\Delta\underline{\alpha}_l \times\right)\Delta\underline{\upsilon}_l$ for which (D-8) becomes the final form

$$\begin{aligned}
\left(\hat{\underline{\eta}}_{l_j} - \underline{\eta}_l\right) &= \frac{1}{2}\left[-\frac{1}{2}\left(\Delta\underline{\alpha}_l \times\right)\right]^j \left(\Delta\underline{\alpha}_l \times\right)\Delta\underline{\upsilon}_l + \frac{1}{2}\left[I + \frac{1}{2}\left(\underline{\phi}_l \times\right)\right]^{-1}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \\
&= -\left[-\frac{1}{2}\left(\Delta\underline{\alpha}_l \times\right)\right]^{j+1} \Delta\underline{\upsilon}_l + \frac{1}{2}\left[I + \frac{1}{2}\left(\underline{\phi}_l \times\right)\right]^{-1}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \\
&\approx -\left[-\frac{1}{2}\left(\Delta\underline{\alpha}_l \times\right)\right]^{j+1} \Delta\underline{\upsilon}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right)
\end{aligned} \tag{D-12}$$

APPENDIX E – DERIVATION OF EQUATION (29)

This appendix derives the error induced in the (15) down-summing algorithm when $\Delta\underline{\alpha}_l$ is approximated by $\underline{\phi}_l$, and both $\underline{\eta}_n$ and $\Delta\underline{\upsilon}_l$ are approximated by $\widehat{\underline{\eta}_n}$, where $\widehat{\underline{\eta}_n}$ is the value estimated by the (22) iteration process for $\underline{\eta}_n$. The main text following (28) shows that after a few iteration of (22), $\widehat{\underline{\eta}_l} = \underline{\eta}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right)$, the value to be used in this appendix.

Paralleling the approach taken in Appendix B, the nominal down-summing formula in (15) is

$$\underline{\eta}_n = \sum_{t_{n-1}}^{t_n}\left[\underline{\eta}_l + \frac{1}{2}\left(\underline{\alpha}_{l-1} \times \Delta\underline{\upsilon}_l - \Delta\underline{\alpha}_l \times \underline{\upsilon}_{l-1}\right)\right] \tag{E-1}$$

With the previously designated approximations, the estimated value for $\underline{\eta}_n$ in (E-1) is

$$\widehat{\underline{\eta}_n} = \sum_{t_{n-1}}^{t_n}\left[\widehat{\underline{\eta}_l} + \frac{1}{2}\left(\widehat{\underline{\alpha}_{l-1}} \times \widehat{\Delta\underline{\upsilon}_l} - \widehat{\Delta\underline{\alpha}_l} \times \widehat{\underline{\upsilon}_{l-1}}\right)\right] \tag{E-2}$$

with

$$\begin{aligned}
\widehat{\Delta\underline{\alpha}_l} &= \underline{\phi}_l = \Delta\underline{\alpha}_l + \delta\underline{\phi}_l \\
\widehat{\underline{\alpha}_{l-1}} &= \sum_{t_{n-1}}^{t_{l-1}} \widehat{\Delta\underline{\alpha}_l} = \sum_{t_{n-1}}^{t_{l-1}} \left(\Delta\underline{\alpha}_l + \delta\underline{\phi}_l\right) = \underline{\alpha}_{l-1} + \sum_{t_{n-1}}^{t_{l-1}} \delta\underline{\phi}_l \\
\widehat{\underline{\eta}_l} &= \underline{\eta}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \\
\widehat{\Delta\underline{\upsilon}_l} &= \widehat{\underline{\eta}_l} = \underline{\eta}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \\
&= \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l + \frac{1}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \approx \Delta\underline{\upsilon}_l + \delta\underline{\eta}_l \\
\widehat{\underline{\upsilon}_{l-1}} &= \sum_{t_{n-1}}^{t_{l-1}} \widehat{\Delta\underline{\upsilon}_l} \approx \sum_{t_{n-1}}^{t_{l-1}} \left(\Delta\underline{\upsilon}_l + \delta\underline{\eta}_l\right) = \underline{\upsilon}_{l-1} + \sum_{t_{n-1}}^{t_{l-1}} \delta\underline{\eta}_l
\end{aligned} \tag{E-3}$$

Substituting from (E-3) as in (B-4) of Appendix B, the $\left(\widehat{\underline{\alpha}_{l-1}} \times \widehat{\Delta\underline{\upsilon}_l} - \widehat{\Delta\underline{\alpha}_l} \times \widehat{\underline{\upsilon}_{l-1}}\right)$ term in (E-2) becomes

$$
\begin{aligned}
&\widehat{\underline{\alpha}_{l-1}} \times \widehat{\Delta \underline{\upsilon}_l} - \widehat{\Delta \underline{\alpha}_l} \times \widehat{\underline{\upsilon}_{l-1}} \\
&= \underline{\alpha}_{l-1} \times \Delta \underline{\upsilon}_l - \Delta \underline{\alpha}_l \times \underline{\upsilon}_{l-1} + \underline{\alpha}_{l-1} \times \delta \underline{\eta}_l - \delta \underline{\phi}_l \times \underline{\upsilon}_{l-1} \\
&\quad + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \right) \times \Delta \underline{\upsilon}_l - \Delta \underline{\alpha}_l \times \sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\eta}_l
\end{aligned}
\tag{E-4}
$$

With (E-4) and $\widehat{\underline{\eta}_l}$ from (E-3), (B-2) becomes

$$
\begin{aligned}
\widehat{\underline{\eta}_n} &= \sum_{t_{n-1}}^{t_n} \left[\widehat{\underline{\eta}_l} + \frac{1}{2} \left(\widehat{\underline{\alpha}_{l-1}} \times \widehat{\Delta \underline{\upsilon}_l} - \widehat{\Delta \underline{\alpha}_l} \times \widehat{\underline{\upsilon}_{l-1}} \right) \right] \\
&= \sum_{t_{n-1}}^{t_n} \left\{ \begin{array}{c} \underline{\eta}_l + \frac{1}{2} \left(\delta \underline{\eta}_l \times \Delta \underline{\alpha}_l - \delta \underline{\phi}_l \times \Delta \underline{\upsilon}_l \right) \\ + \frac{1}{2} \left[\begin{array}{c} \underline{\alpha}_{l-1} \times \Delta \underline{\upsilon}_l - \Delta \underline{\alpha}_l \times \underline{\upsilon}_{l-1} + \left(\sum_{t_{n-1}}^{t_{l-1}} \Delta \underline{\alpha} \right) \times \delta \underline{\eta}_l - \delta \underline{\phi}_l \times \underline{\upsilon}_{l-1} \\ + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \right) \times \Delta \underline{\upsilon}_l - \Delta \underline{\alpha}_l \times \sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\eta}_l \end{array} \right] \end{array} \right\} \\
&= \underline{\eta}_n + \frac{1}{2} \sum_{t_{n-1}}^{t_n} \left[\begin{array}{c} \left(\delta \underline{\eta}_l \times \Delta \underline{\alpha}_l - \delta \underline{\phi}_l \times \Delta \underline{\upsilon}_l \right) \\ + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \right) \times \Delta \underline{\upsilon}_l - \delta \underline{\phi}_l \times \sum_{t_{n-1}}^{t_{l-1}} \Delta \underline{\upsilon} + \left(\sum_{t_{n-1}}^{t_{l-1}} \Delta \underline{\alpha} \right) \times \delta \underline{\eta}_l - \Delta \underline{\alpha}_l \times \sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\eta}_l \end{array} \right]
\end{aligned}
\tag{E-5}
$$

Thus,

$$
\begin{aligned}
&\widehat{\underline{\eta}_n} - \underline{\eta}_n = \frac{1}{2} \sum_{t_{n-1}}^{t_n} \left(\delta \underline{\eta}_l \times \Delta \underline{\alpha}_l - \delta \underline{\phi}_l \times \Delta \underline{\upsilon}_l \right) \\
&+ \frac{1}{2} \sum_{t_{n-1}}^{t_n} \left[\underline{\alpha}_{l-1} \times \delta \underline{\eta}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\eta}_l \right) \times \Delta \underline{\alpha}_l + \underline{\upsilon}_{l-1} \times \delta \underline{\phi}_l + \left(\sum_{t_{n-1}}^{t_{l-1}} \delta \underline{\phi}_l \right) \times \Delta \underline{\upsilon}_l \right]
\end{aligned}
\tag{E-6}
$$

APPENDIX F – DERIVATION OF EQ. (30)

This appendix derives analytical values for (E-6) of Appendix E when the $\delta \underline{\phi}_l$, $\delta \underline{\eta}_l$ coning/sculling corrections are constant, and the $\Delta \underline{\alpha}_l$, $\Delta \underline{\upsilon}_l$ integrated angular-rate/specific-force increments are characterized by a constant plus a linear ramping increase in time as in (C-1) of Appendix C:

$$
\Delta \underline{\alpha}_l = \Delta \underline{\alpha}_c + \Delta \underline{\alpha}_{rmp}\, l \qquad \Delta \underline{\upsilon}_l = \Delta \underline{\upsilon}_c + \Delta \underline{\upsilon}_{rmp}\, l
\tag{F-1}
$$

Using (F-1), the $\frac{1}{2}\sum_{t_{n-1}}^{t_n}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right)$ term in (E-6) becomes

$$
\begin{aligned}
\frac{1}{2}\sum_{t_{n-1}}^{t_n}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) &= \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\delta\underline{\eta}_l \times \left(\Delta\underline{\alpha}_c + \Delta\underline{\alpha}_{rmp}\, l\right) - \delta\underline{\phi}_l \times \left(\Delta\underline{\upsilon}_c + \Delta\underline{\upsilon}_{rmp}\, l\right)\right] \\
&= \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_c - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_c\right) + \frac{1}{2}\sum_{t_{n-1}}^{t_n}\left[\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp} - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp}\right) l\right] \\
&= \frac{L}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_c - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_c\right) + \frac{1}{2}\left(\sum_{l=1}^{L} l\right)\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp} - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp}\right)
\end{aligned}
\tag{F-2}
$$

As in Appendix C, the $\sum_{l=1}^{L} l$ term in (F-1) is the sum of an L term arithmetic progression in l starting with 1 and ending in L which from [20, Sect. I - ALGEBRA, Subsect. 10] equals $L(1+L)/2$. Thus,

$$
\begin{aligned}
&\frac{1}{2}\sum_{t_{n-1}}^{t_n}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_l - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_l\right) \\
&= \frac{L}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_c - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_c\right) + \frac{L(1+L)}{4}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp} - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp}\right)
\end{aligned}
\tag{F-3}
$$

Eq. (F-3) represents the first term in (E-6) under the (F-1) conditions. The second term in (F-3) is given by (C-5) in Appendix C. Thus, the total (F-3) response is

$$
\begin{aligned}
\widehat{\underline{\eta}_n} - \underline{\eta}_n &= \frac{L}{2}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_c - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_c\right) + \frac{L(1+L)}{4}\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp} - \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp}\right) \\
&= \frac{1}{4}\left(\sum_{l=1}^{L} l\,(l-1)\right)\left(\delta\underline{\eta}_l \times \Delta\underline{\alpha}_{rmp} + \delta\underline{\phi}_l \times \Delta\underline{\upsilon}_{rmp}\right)
\end{aligned}
\tag{F-4}
$$

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