IMPROVED STRAPDOWN INERTIAL SYSTEM CALIBRATION PROCEDURES

PART 2 - ANALYTICAL DERIVATIONS

Paul G. Savage
Strapdown Associates, Inc.
Maple Plain, MN 55359 USA

WBN-14020-2
www.strapdowassociates.com
October 20, 2017 (Updated January 11, 2018)

ABSTRACT

This article is Part 2 of a three part series describing an improved Strapdown Rotation Test (SRT) for calibrating the compensation coefficients in a strapdown inertial measurement unit (IMU). The SRT consists of a set of IMU rotations and processing routines that enable precision measurements of IMU gyro/accelerometer misalignment and scale factor calibration errors, and accelerometer bias calibration errors, all without precision rotation fixturing. The improved SRT is compatible with a broad range of IMU types from aircraft accuracy inertial navigation systems (INSs) to the latest low cost MEMS variety (Micro-machined Electronic Module System). This Part 2 article derives the Part 1 equations for 1) Data collection during rotation tests, post-test data processing, 2) Determining sensor errors from recorded test data, 3) Calculating errors in determining sensor errors due to rotation fixture inaccuracy, 4) Calculating errors in determining sensor errors due to misalignment in mounting the system on the rotation fixture, 5) Calculating errors in determining sensor errors due approximations in post-test data analysis equations, and 6) Correcting sensor error determinations for the effect of gyro bias residuals during the SRT.

FOREWORD

This article is the second in a three part series describing improved strapdown rotation test (SRT) procedures for calibrating a strapdown inertial measurement unit (IMU) containing an orthogonal triad of inertial sensors (gyros and accelerometers), digital processor, associated sensor calibration software, and other computational elements. The improved rotation tests consist of a series of rotation sequences, each designed to measure one of the following errors in sensor calibration coefficients: gyro-to-gyro misalignment, accelerometer-to-gyro misalignment, gyro/accelerometer scale-factor, and accelerometer bias. The first and third articles in the three-part series cover the following topics:

Part 1: Procedures, Rotation Fixtures, And Accuracy Analysis - Describes the general theory for the improved rotation tests, rotation test operations, data collection during test, post-test data processing, rotation test fixture requirements, rotation design for sensor error determination, and sensor error determination accuracy analysis.

Part 3 - Numerical Examples - Provides numerical examples showing how collected SRT rotation test data translates sensor errors into data collection measurements, and the impact of neglecting gyro bias in the SRT sensor error determination process. The results numerically confirm that rotation sequences designed in Part 1 measure the particular sensor error for which they were designed.

1.0 INTRODUCTION

A strapdown IMU (inertial measurement unit) contains inertial instruments (gyros and accelerometers) and associated digital computational electronics for compensating the inertial sensors. Some IMUs (e.g., inertial navigation systems – INSs) also compute attitude by a gyro integration process, use the attitude and accelerometer outputs to calculate acceleration in an angular rate stabilized coordinate frame (similar to the accelerations that would be measured by accelerometers mounted on a gyro stabilized gimbaled platform), and integrate the stabilized accelerations to calculate velocity and position. The Strapdown Rotation Test (SRT) is the method used by many aerospace groups for precision calibrating the misalignments between the gyros and accelerometers in a strapdown measurement unit (IMU). An SRT consists of a sequence of rotation sequences designed to excite IMU sensor errors on IMU outputs. SRT processing uses IMU response measurements to calculate the IMU sensor errors producing them. The results are then used to recalibrate the IMU sensor compensation coefficients to eliminate the sensor errors during subsequent IMU usage.

The principle advantage of the SRT is its ability to precision calibrate sensor alignments (to micro-radian accuracy for an INS) without requiring precision (and expensive) rotation test fixturing and IMU setup time. Secondarily, the SRT also calibrates inertial sensor scale factors and accelerometer biases. The original SRT (disclosed in 1977 [1] and updated in 2000 [2, Sect. 18.4]) was designed primarily for aircraft INS accuracy application, and required an inertial self-alignment of the IMU attitude (relative to local north and vertical) prior to SRT rotation sequence execution. SRT improvements since 2000 have eliminated the initial self-alignment requirement and broadened the IMU applicability range from aircraft accuracy INSs to lower performance micro-machined electronic systems (MEMS) IMUs. The improved SRT is described in detail in Part 1 [3] including an error analysis demonstrating its ability to achieve micro-radian accuracy using one milli-radian accuracy rotation test equipment/procedures. This Part 2 article analytically derives the improved SRT processing equations and associated error models used in Part 1 [3]. The analytical developments are presented in detail, emphasizing the impact of second order effects on results obtained, including the effect of IMU mounting error on the rotation test fixture and rotation fixture incaccuracy in executing SRT rotations. (The error analysis in [3] evaluates the impact of neglected second order effects in SRT processing, demonstrating micro-radian accuracy of the SRT procedure.)

Sections 2.0 and 3.0 of this article describes analytical notation and coordinate frames utilized throughout. Second 4.0 describes the fundamental SRT analytical concept: Computed

IMU accelerations in an inertially stabilized coordinate frame should be zero for a perfectly calibrated IMU, thus non-zero IMU stabilized accelerations are measures of IMU sensor error. This fundamental concept is the basis in Section 4.0 for analytically deriving the SRT measurement processing equations. Section 5.0 then derives second order models for the Section 4.0 SRT measurement equations. Section 6.0 generates a linearized simplified version of the Section 5.0 for use in determining sensor errors from SRT measurements. The Section 7.0 subtracts the simplified Section 6.0 SRT measurement model from the general Section 5.0 model to derive an error model for the SRT sensor error determination process. Section 8.0 provides an assessment of the effect of neglecting residual gyro bias calibration error in the Section 6.0 SRT processing equations. If significant, Part 1 [3, Section 4.8] describes a simple method for mitigating the gyro residual bias impact on SRT accuracy. Appendices A and B describe gyro/accelerometer error and compensation models used in Sections 4.0 – 6.0, and for the compensation equations presented in Part 1 [3, Sect. 4.3.1].

2.0 NOTATION

The following general notation was used throughout this article.

$\underline{V}$ = Vector without specific coordinate frame designation. A vector is a parameter that has length and direction. Vectors used in the paper are classified as "free vectors", hence, have no preferred location in coordinate frames in which they are analytically described.

$\underline{V}^A$ = Column matrix with elements equal to the projection of $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit vector parallel to that coordinate axis.

$\left(\underline{V}^A \times\right)$ = Skew symmetric (or cross-product) form of $\underline{V}^A$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{XA} , V_{YA}, V_{ZA} are the components of $\underline{V}^A$. The matrix product of $\left(\underline{V}^A \times\right)$ with another A frame vector equals the cross-product of $\underline{V}^A$ with the vector in the A frame, i.e.: $\left(\underline{V}^A \times\right) \underline{W}^A = \underline{V}^A \times \underline{W}^A$.

C_{A2}^{A1} = Direction cosine matrix that transforms a vector from its coordinate frame A_2 projection form to its coordinate frame A_1 projection form, i.e., $\underline{V}^{A1} = C_{A2}^{A1} \, \underline{V}^{A2}$. The columns of C_{A2}^{A1} are projections on A_1 axes of unit vectors parallel to A_2 axes.

Conversely, the rows of $C_{A_2}^{A_1}$ are projections on A_2 axes of unit vectors parallel to A_1 axes. An important property of $C_{A_2}^{A_1}$ is that it's inverse equals it's transpose.

$\underline{\omega}_{I:A}$ = Angular rotation rate of generalized coordinate frame *A* relative to inertially non-rotating space (*I : A* subscript).

$\underline{\omega}_{I:E}$ = Angular rotation rate of the earth relative to inertially non-rotating space (*I : E* subscript).

$\underline{\omega}_{E:A}$ = Angular rotation rate of generalized coordinate frame *A* relative to the rotating earth (*E : A* subscript). Note that $\underline{\omega}_{I:A} = \underline{\omega}_{I:E} + \underline{\omega}_{E:A}$ and equivalently, $\underline{\omega}_{E:A} = \underline{\omega}_{I:A} - \underline{\omega}_{I:E}$.

$\dot{(\,)} = \dfrac{d(\,)}{dt}$ = Derivative of parameter ( ) with respect to time *t*.

$\widehat{(\,)}$ = Computed or measured value of parameter ( ) that, in contrast with the idealized error free value ( ), contains errors.

3.0 COORDINATE FRAMES

Coordinate frames used in this article can be grouped into two types; Basic and Refined.

3.1 BASIC COORDINATE FRAMES

The primary coordinate frame used in this article is the IMU fixed *B* frame that is rotated relative to the earth (and inertial space) during each SRT rotation sequence. Other coordinate frames related to *B* are fixed (non-rotating) relative to the earth, most aligned with the *B* frame at the start and end of a rotation sequence, one defined to be aligned with north, east, down coordinates at the test site. Specific definitions for the coordinate frame are as follows:

B = IMU sensor frame that is fixed relative to strapdown inertial sensor input axes, but that rotates relative to the earth during each rotation sequence of the SRT. The angular orientation of the *B* frame relative to sensor axes is arbitrary based on user or traditional preferences.

MARS = Designation for a "mean-angular-rate-sensor" *B* frame selection, the orthogonal frame that best fits around the actual strapdown gyro input axes.

NED = Earth fixed coordinate frame having axes aligned with local north, east, down directions.

B_{Strt} = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the B frame at the start of the rotation sequence. Nominally, one of the B_{Strt} frame axes would be aligned with the local vertical if the IMU being tested was perfectly mounted on an idealized rotation fixture.

B_{End} = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the B frame at the end of the rotation sequence.

$B_{i,\,Strt}$ = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the B frame at the start of rotation i in a rotation sequence.

$B_{i,End}$ = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the B frame at the end of rotation i in a rotation sequence.

3.2 REFINED COORDINATE FRAMES

In addition to the basic coordinate frames described in Section 3.1, the concept of "nominal" B frame coordinates is used to describe angular motion of an IMU under test having an idealized (error free) mounting on an idealized rotation-fixture that can execute prescribed rotations without error. Analogous to the B frame, the nominal B frame (B^{Nom}) rotates relative to the earth (and inertial space) during rotation segments of each rotation sequence. All other nominal coordinate frames are fixed (non-rotating) relative to the earth, most defined to be aligned with B^{Nom} at the start and end of a rotation sequence, one defined to be aligned with north, east, down coordinates at the test site. Specific definitions for the nominal coordinate frame are as follows:

B^{Nom} = Nominal B frame defined as a hypothetical B frame that is nominally mounted on a nominal idealized rotation fixture that executes rotations exactly as prescribed, and which was installed in the test facility exactly as prescribed relative to local *NED* coordinates (i.e., so that the orientation of the B_{Nom} frame is known without error for any commanded rotation fixture gimbal angles).

B_{Strt}^{Nom} = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the B^{Nom} frame at the start of the rotation sequence. Nominally, one of the B_{Strt}^{Nom} frame axes (x, y, or z) would be aligned with the local vertical if the inertial measurement unit (IMU being rotation tested is perfectly mounted on an idealized rotation fixture.

B_{End}^{Nom} = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the B^{Nom} frame at the end of the rotation sequence.

$B_{i,\,Strt}^{Nom}$ = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the B^{Nom} frame at the start of rotation i in a rotation sequence.

$B_{i,\,End}^{Nom}$ = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the B^{Nom} frame at the end of rotation i in a rotation sequence.

4.0 ANALYTICAL BASIS FOR THE IMPROVED STRAPDOWN ROTATION TEST

The analytical basis for the Strapdown Rotation Test (SRT) derives from the fundamental identity that for a stationary IMU, the true acceleration relative to the earth is zero:

$$\underline{a}^{B} = \underline{a}_{SF}^{B} + g\,\underline{u}_{Dwn}^{B} = 0 \tag{1}$$

where

$\underline{a}^{B}$ = True IMU acceleration vector relative to the earth in general B frame coordinates <u>when the IMU is stationary</u>.

$\underline{a}_{SF}^{B}$ = True specific force acceleration vector relative to the earth (in general B frame coordinates), the acceleration sensed by the IMU accelerometers.

$\underline{u}_{Dwn}^{B}$ = True unit vector downward at the test site (parallel to plumb-bob gravity) in general B frame coordinates.

g = Plumb-bob gravity magnitude at the test site.

The $\underline{u}_{Dwn}^{B}$ vector in (1) can be calculated as

$$\underline{u}_{Dwn}^{B} = C_{NED}^{B}\,\underline{u}_{Dwn}^{NED} \qquad \underline{u}_{Dwn}^{NED} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{T} \tag{2}$$

where

C_{NED}^{B} = Direction cosine matrix that transforms vectors from NED to B frame coordinates.

With (2), (1) transformed to the B frame at the start of the rotation sequence becomes

$$\underline{a}^{B\,Strt} = C_{B}^{B\,Strt}\,\underline{a}_{SF}^{B} + g\,C_{NED}^{B\,Strt}\,\underline{u}_{Dwn}^{NED} = 0 \tag{3}$$

Equation (3) is valid for any stationary B frame orientation, thus, at the start and end of the rotation sequence,

$$\underline{a}_{Strt}^{B\,Strt} \equiv \left(\underline{a}^{B\,Strt}\right)_{Strt} = 0 \qquad \underline{a}_{End}^{B\,Strt} \equiv \left(\underline{a}^{B\,Strt}\right)_{End} = 0 \tag{4}$$

where

$\underline{a}_{Strt}^{B\,Strt}$, $\underline{a}_{End}^{B\,Strt}$ = $\underline{a}^{B\,Strt}$ in (3) measured at the start and end of the rotation sequence.

4.1 IMPROVED STRAPDOWN ROTATION TEST MEASUREMENTS

Equations (4) with (3) are the basis for the SRT measurement concept. For a perfect (error free) IMU and a known B_{Strt} frame orientation relative to *NED* coordinates, $\underline{a}_{Strt}^{B\,Strt}$ and $\underline{a}_{End}^{B\,Strt}$ in (4) are zero. Therefore, non-zero values for values for $\underline{a}_{Strt}^{B\,Strt}$ and $\underline{a}_{End}^{B\,Strt}$ represent measurements of IMU and test execution error. The equivalent of (3) and (4) using actual data (containing errors) would then be

$$\begin{gathered} \hat{\underline{a}}^{B\,Strt} = \hat{C}_B^{B\,Strt}\, \hat{\underline{a}}_{SF}^{B} + g\, \hat{C}_{NED}^{B\,Strt}\, \underline{u}_{Dwn}^{NED} \\ \hat{\underline{a}}_{Strt}^{B\,Strt} \equiv \left(\hat{\underline{a}}^{B\,Strt}\right)_{Strt} \neq 0 \qquad \hat{\underline{a}}_{End}^{B\,Strt} \equiv \left(\hat{\underline{a}}^{B\,Strt}\right)_{End} \neq 0 \end{gathered} \tag{5}$$

where

$\widehat{(\,)}$ = Parameter ( ) containing errors (in contrast with parameter ( ) defined to be error free).

$\left(\hat{\underline{a}}^{B\,Strt}\right)_{Strt}$, $\left(\hat{\underline{a}}^{B\,Strt}\right)_{End}$ = $\hat{\underline{a}}^{B\,Strt}$ at the start and end of the rotation sequence.

$\hat{\underline{a}}_{SF}^{B}$ = Compensated accelerometer specific force vector, the output of the Part 1 [3, Eqs. (2)] accelerometer compensation equations that still contain residual errors (to be determined by the SRT).

$\hat{C}_B^{B\,Strt}$ = Direction cosine matrix that transforms vectors from the B frame to the B_{Strt} frame. The $\hat{C}_B^{B\,Strt}$ matrix is calculated using angular rate outputs the Part 1 [3, Eqs. (2)] gyro compensation equations that still contain residual errors (to be determined by the SRT).

Non-zero values for $\hat{\underline{a}}_{Strt}^{B\,Strt}$ and $\hat{\underline{a}}_{End}^{B\,Strt}$ in (5) are due to residual errors in $\hat{C}_B^{B\,Strt}$, $\hat{\underline{a}}_{SF}^{B}$, and $\hat{C}_{NED}^{B\,Strt}$. Using an error model for $\hat{\underline{a}}_{Strt}^{B\,Strt}$ and $\hat{\underline{a}}_{End}^{B\,Strt}$ allows the accelerometer and gyro residual compensation errors to be determined using classical matrix inversion methods.

From (5) it can be seen that $\hat{C}_{NED}^{B\,Strt}\,\underline{u}_{Dwn}^{NED}$ is contained in both $\hat{\underline{a}}_{Strt}^{B\,Strt}$ and $\hat{\underline{a}}_{End}^{B\,Strt}$. To eliminate this term (and $\hat{C}_B^{B\,Strt}$ initialization error) from the error determination process, the difference between $\hat{\underline{a}}_{Strt}^{B\,Strt}$ and $\hat{\underline{a}}_{End}^{B\,Strt}$ is used for horizontal SRT measurements:

$$\hat{\underline{a}}_{SF}^{B\,Strt} = \hat{C}_B^{B\,Strt}\,\hat{\underline{a}}_{SF}^{B}$$
$$\Delta\hat{\underline{a}}^{B\,Strt} \equiv \hat{\underline{a}}_{End}^{B\,Strt} - \hat{\underline{a}}_{Strt}^{B\,Strt} = \left(\hat{\underline{a}}_{SF}^{B\,Strt}\right)_{End} - \left(\hat{\underline{a}}_{SF}^{B\,Strt}\right)_{Strt} = \hat{\underline{a}}_{SF\,Strt}^{B\,Strt} - \hat{\underline{a}}_{SF\,End}^{B\,Strt} \tag{6}$$

where

$\hat{\underline{a}}_{SF\,Strt}^{B\,Strt}$, $\hat{\underline{a}}_{SF\,End}^{B\,Strt}$ = $\hat{\underline{a}}_{SF}^{B\,Strt}$ at the start and end of the rotation sequence (subscripts) in B_{Strt} coordinates (superscript).

$\Delta\hat{\underline{a}}^{B\,Strt}$ = Difference between stationary IMU acceleration vectors at the start and end of the rotation sequence in B_{Strt} coordinates (used for the horizontal SRT measurement component).

Note: Eq. (6) in the form shown is somewhat symbolic. To account for IMU micro-movement and to minimize sensor quantization error effects, $\hat{\underline{a}}_{SF\,Strt}^{B\,Strt}$ in (6) would actually be calculated as an average of $\hat{C}_B^{B\,Strt}\,\hat{\underline{a}}_{SF}^{B}$ immediately before the rotation sequence is executed. Similarly, $\hat{\underline{a}}_{SF\,End}^{B\,Strt}$ in (6) would be calculated as an average of $\hat{C}_B^{B\,Strt}\,\hat{\underline{a}}_{SF}^{B}$ immediately following completion of the rotation sequence.

As mentioned previously, an important characteristic of (6) is that because $\hat{C}_B^{B\,Strt}$ is generated by an integration process (described subsequently), the $\hat{C}_B^{B\,Strt}$ initialization error in $\hat{\underline{a}}_{End}^{B\,Strt}$ and $\hat{\underline{a}}_{Strt}^{B\,Strt}$ is cancelled (to first order accuracy) in making the $\Delta\hat{\underline{a}}^{B\,Strt}$ difference measurement operation. The result is simplification in IMU mounting and rotation fixture alignment accuracy requirements. (Note: The original SRT [1] was designed for aircraft INS sensor calibration, and used the INS gyro and accelerometer signals to implement an inertial self-alignment process for $\hat{C}_B^{B_{Strt}}$ initialization. This method is not practical for IMUs having lesser accuracy inertial sensors – e.g., MEMS based).

The horizontal component of $\Delta\underline{\hat{a}}^{B_{Strt}}$ in (6) used for the SRT horizontal measurement as analytically defined by

$$\Delta\underline{\hat{a}}_H^{B_{Strt}} = \left(\underline{\hat{a}}_{SF_{End}}^{B_{Strt}} - \underline{\hat{a}}_{SF_{Strt}}^{B_{Strt}}\right)_H \equiv \left[I - \underline{\hat{u}}_{Dwn}^{B_{Strt}}\left(\underline{\hat{u}}_{Dwn}^{B_{Strt}} \cdot\right)\right]\left(\underline{\hat{a}}_{SF_{End}}^{B_{Strt}} - \underline{\hat{a}}_{SF_{Strt}}^{B_{Strt}}\right) \quad (7)$$

where

H = Designation of a vector's horizontal component defined as the vector minus its vertical component.

$\underline{\hat{u}}_{Dwn}^{B_{Strt}}$ = Unit vector downward in B_{Strt} frame coordinates (and containing error).

$\left(\underline{\hat{u}}_{Dwn}^{B_{Strt}} \cdot\right)$ = $\underline{\hat{u}}_{Dwn}^{B_{Strt}}$ dot product operator. When $\underline{\hat{u}}_{Dwn}^{B_{Strt}}$ is represented as a column matrix, $\left(\underline{\hat{u}}_{Dwn}^{B_{Strt}} \cdot\right)$ is the transpose of $\underline{\hat{u}}_{Dwn}^{B_{Strt}}$.

I = Identity matrix.

$\Delta\underline{\hat{a}}_H^{B_{Strt}}$ = Horizontal component of $\Delta\underline{\hat{a}}^{B_{Strt}}$ in (6).

The $\underline{\hat{u}}_{Dwn}^{B_{Strt}}$ vector in (7) is from the (2) equivalent in B_{Strt} coordinates:

$$\underline{\hat{u}}_{Dwn}^{B_{Strt}} = \hat{C}_{NED}^{B_{Strt}}\, \underline{u}_{Dwn}^{NED} \qquad \underline{u}_{Dwn}^{NED} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \quad (8)$$

To go with (7) - (8), the vertical components of $\underline{\hat{a}}_{Strt}^{B_{Strt}}$ and $\underline{\hat{a}}_{End}^{B_{Strt}}$ in (5) are calculated as

$$\hat{a}_{Strt_{Dwn}}^{B_{Strt}} = \underline{\hat{u}}_{Dwn}^{B_{Strt}} \cdot \underline{\hat{a}}_{Strt}^{B_{Strt}} \qquad \hat{a}_{End_{Dwn}}^{B_{Strt}} = \underline{\hat{u}}_{Dwn}^{B_{Strt}} \cdot \underline{\hat{a}}_{End}^{B_{Strt}} \quad (9)$$

where

$\hat{a}_{Strt_{Dwn}}^{B_{Strt}}, \hat{a}_{End_{Dwn}}^{B_{Strt}}$ = Downward components of $\underline{\hat{a}}_{Strt}^{B_{Strt}}$ and $\underline{\hat{a}}_{End}^{B_{Strt}}$ in (5).

The $\hat{\underline{a}}_{Strt}^{B_{Strt}}$ and $\hat{\underline{a}}_{End}^{B_{Strt}}$ vectors in (9) are derived from (5) with the general B designation equated to B_{Strt} and B_{End}, while recognizing that $\hat{C}_{NED}^{B_{Strt}} \underline{u}_{Dwn}^{NED} = \hat{\underline{u}}_{Dwn}^{B_{Strt}}$:

$$\hat{\underline{a}}_{Strt}^{B_{Strt}} = \hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}} + g\, \hat{\underline{u}}_{Dwn}^{B_{Strt}} \qquad \hat{\underline{a}}_{End}^{B_{Strt}} = \hat{\underline{a}}_{SF_{End}}^{B_{Strt}} + g\, \hat{\underline{u}}_{Dwn}^{B_{Strt}} \tag{10}$$

Substituting (10) into (9) then finds

$$\hat{a}_{Strt_{Dwn}}^{B_{Strt}} = \hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}} + g \qquad \hat{a}_{End_{Dwn}}^{B_{Strt}} = \hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \left(\hat{\underline{a}}_{SF_{End}}^{B_{Strt}} \right) + g \tag{11}$$

It remains to define equations for $\hat{C}_B^{B_{Strt}}$ in (5) and (6). A key element in the SRT process calculates $\hat{C}_B^{B_{Strt}}$ as an integration process using gyro data measurements for input. This enables the $\Delta\hat{\underline{a}}_H^{B_{Strt}}$ measurement to include the effect of gyro error, deeming it deterministic during post-processing calculations. Additionally and importantly (as will be apparent subsequently), use of a gyro based $\hat{C}_B^{B_{Strt}}$ determination enables the accelerometer alignment error contribution in $\Delta\hat{\underline{a}}_H^{B_{Strt}}$ to be a direct function of relative misalignment between the gyro and accelerometer input axes, the misalignment factor that impacts IMU navigation accuracy. Finally and also importantly (as mentioned previously), use of a gyro data computed $\hat{C}_B^{B_{Strt}}$ in (5) - (6) eliminates the need for precise alignment of the IMU to the rotation fixture and local vertical during the SRT. The result is both improved sensor error determination accuracy and reduced rotation test fixture accuracy/cost requirements.

The computation for $\hat{C}_B^{B_{Strt}}$ in (7) is based on the exact $C_B^{B_{Strt}}$ direction cosine rate equation [2, Eq. (4.1-1)], while recognizing that by definition, $C_B^{B_{Strt}}$ is identity at rotation sequence start:

$$C_B^{B_{Strt}} = I + \int_{t_{Strt}}^{t} \dot{C}_B^{B_{Strt}}\, dt \qquad \dot{C}_B^{B_{Strt}} = C_B^{B_{Strt}} \left(\underline{\omega}_{I:B}^{B} \times \right) - \left(\underline{\omega}_{I:E}^{B_{Strt}} \times \right) C_B^{B_{Strt}} \tag{12}$$

in which

$$\underline{\omega}_{I:E}^{B_{Strt}} = C_{NED}^{B_{Strt}}\, \underline{\omega}_{I:E}^{NED} \qquad \underline{\omega}_{I:E}^{NED} = \left[\omega_e \cos l \quad 0 \quad -\omega_e \sin l \right]^T \tag{13}$$

where

$\underline{\omega}_{I:B}^{B}$ = Angular rate vector of the B frame relative to non-rotating inertial space ($I:B$ subscript) in B frame coordinates (superscript), the angular rate vector sensed by the IMU strapdown gyro triad.

$\underline{\omega}_{I:E}^{B_{Strt}}$, $\underline{\omega}_{I:E}^{NED}$ = Angular rate vector of the earth relative to non-rotating inertial space ($I:E$ subscript) in B_{Strt} and NED frame coordinates (superscripts).

ω_e = Earth's rotation rate magnitude relative to non-rotating inertial space.

l = Latitude of the test site.

For the SRT, the (12) and (13) calculations would be performed using actual gyro data (the output of the Part 1 [3, Eqs. (2)] compensation equations that still contain residual errors), estimated (uncertain) values of $C_{NED}^{B_{Strt}}$, and an assumed ideal alignment of the B frame with the B_{Strt} frame, i.e., $\left(\hat{C}_B^{B_{Strt}}\right)_{Strt}$ = Identity :

$$\hat{C}_B^{B_{Strt}} = I + \int_{t_{SeqStrt}}^{t} \dot{\hat{C}}_B^{B_{Strt}} dt$$

$$\dot{\hat{C}}_B^{B_{Strt}} = \hat{C}_B^{B_{Strt}}\left(\underline{\hat{\omega}}_{IB}^{B}\times\right) - \left(\underline{\hat{\omega}}_{IE}^{B_{Strt}}\times\right)\hat{C}_B^{B_{Strt}} \tag{14}$$

$$\underline{\hat{\omega}}_{IE}^{B_{Strt}} = \hat{C}_{NED}^{B_{Strt}}\,\underline{\omega}_{IE}^{NED} \qquad \underline{\omega}_{I:E}^{NED} = \left[\omega_e \cos l \quad 0 \quad -\omega_e \sin l\right]^T$$

To summarize, SRT measurements taken for each rotation sequence would be $\Delta\underline{\hat{a}}_H^{B_{Strt}}$, $\hat{a}_{Strt_{Dwn}}^{B_{Strt}}$, and $\hat{a}_{End_{Dwn}}^{B_{Strt}}$ calculated with (6) - (8), (11), and (14) as listed next.

$$\underline{\hat{a}}_{SF}^{B_{Strt}} = \hat{C}_B^{B_{Strt}}\,\underline{\hat{a}}_{SF}^{B}$$

$$\underline{\hat{a}}_{SF_{Strt}}^{B_{Strt}} \equiv \left(\underline{\hat{a}}_{SF}^{B_{Strt}}\right)_{Strt} \qquad \underline{\hat{a}}_{SF_{End}}^{B_{Strt}} \equiv \left(\underline{\hat{a}}_{SF}^{B_{Strt}}\right)_{End}$$

$$\Delta\underline{\hat{a}}_H^{B_{Strt}} = \left(\underline{\hat{a}}_{SF_{End}}^{B_{Strt}} - \underline{\hat{a}}_{SF_{Strt}}^{B_{Strt}}\right)_H \tag{15}$$

$$\underline{\hat{u}}_{Dwn}^{B_{Strt}} = \hat{C}_{NED}^{B_{Strt}}\,\underline{u}_{Dwn}^{NED} \qquad \underline{u}_{Dwn}^{NED} = \left[0 \quad 0 \quad 1\right]^T$$

$$\hat{a}_{Strt_{Down}}^{B_{Strt}} = \underline{\hat{u}}_{Dwn}^{B_{Strt}} \cdot \underline{\hat{a}}_{SF_{Strt}}^{B_{Strt}} + g \qquad \hat{a}_{End_{Down}}^{B_{Strt}} = \underline{\hat{u}}_{Dwn}^{B_{Strt}} \cdot \left(\underline{\hat{a}}_{SF_{End}}^{B_{Strt}}\right) + g$$

$$\hat{C}_B^{B_{Strt}} = I + \int_{t_{SeqStrt}}^{t} \dot{\hat{C}}_B^{B_{Strt}} dt$$

$$\dot{\hat{C}}_B^{B_{Strt}} = \hat{C}_B^{B_{Strt}}\left(\underline{\hat{\omega}}_{IB}^{B}\times\right) - \left(\underline{\hat{\omega}}_{IE}^{B_{Strt}}\times\right)\hat{C}_B^{B_{Strt}} \tag{16}$$

$$\underline{\hat{\omega}}_{IE}^{B_{Strt}} = \hat{C}_{NED}^{B_{Strt}}\,\underline{\omega}_{IE}^{NED} \qquad \underline{\omega}_{I:E}^{NED} = \left[\omega_e \cos l \quad 0 \quad -\omega_e \sin l\right]^T$$

The $\hat{C}_{NED}^{B_{Strt}}$ matrix in (15) would be estimated for each rotation sequence from the approximately known angular orientation of the IMU mounted on the rotation test at the test site. The $\hat{\underline{a}}_{SF_{End}}^{B_{End}}$, $\hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}}$ terms in (16) would be calculated as the average $\hat{\underline{a}}_{SF}$ outputs from the Part 1 [3, Eqs. (2)] accelerometer compensation equations.

Equations (15) - (16) are applied for each rotation sequence in the SRT to generate a set of $\Delta\hat{\underline{a}}_{H}^{B_{Strt}}$, $\hat{a}_{Strt_{Dwn}}^{B_{Strt}}$, $\hat{a}_{End_{Dwn}}^{B_{Strt}}$ measurements. By appropriately structuring the rotation sequences (as in Part 1 [3, Table 1]), a deterministic set of $\Delta\hat{\underline{a}}_{H}^{B_{Strt}}$, $\hat{a}_{Strt_{Dwn}}^{B_{Strt}}$, and $\hat{a}_{End_{Dwn}}^{B_{Strt}}$ equations is established as a function of Part 1 [3, Eqs. (2)] compensation coefficient error residuals. Analytical inversion of the equations allows the error residuals to be calculated, then applied as corrections (calibration) to the Part 1 [3, Eqs. (2)] compensation coefficients.

5.0 MEASUREMENTS IN TERMS OF SENSOR ERRORS

This section derives second order analytical model equivalents to SRT measurement Eqs. (15) - (16) as functions of the sensor and IMU mounting/rotation errors that generate non-zero values for the (15) stationary acceleration measurements. The results will be used to formulate simplified linearized models used by the SRT for deducing sensor errors from the (15) measurements, and to formulate second order error equations for evaluating SRT accuracy in sensor error determination. To analytically account for uncertainties in rotation fixture installation at the test site, IMU installation uncertainty on the rotation test fixture, and rotation fixture inaccuracies in executing prescribed rotations, the Section 3.2 refined coordinate frame definitions will be applied in the derivation process.

Analytical model derivations begin with the Eq. (1) fundamental SRT truth model principle that for any B frame orientation, IMU acceleration under stationary conditions is zero. At the start and end of a particular SRT rotation sequence, (1) becomes

$$\underline{a}_{Strt}^{B} = \underline{a}_{SF_{Strt}}^{B} + g\,\underline{u}_{Dwn}^{B} = 0 \qquad \underline{a}_{End}^{B} = \underline{a}_{SF_{End}}^{B} + g\,\underline{u}_{Dwn}^{B} = 0 \tag{17}$$

where

$\underline{a}_{Strt}^{B}$, $\underline{a}_{End}^{B}$, $\underline{a}_{SF_{Strt}}^{B}$, $\underline{a}_{SF_{End}}^{B}$ = $\underline{a}^{B}$, $\underline{a}_{SF}^{B}$ at the start and end of the rotation sequence, projected on arbitrary B frame orientation axes.

From (17), $\underline{a}_{End}^{B}$ and $\underline{a}_{End}^{B}$ projected on B_{Strt} frame coordinate axes is

$$
\begin{aligned}
\underline{a}_{Strt}^{B\,Strt} &= \underline{a}_{SF\,Strt}^{B\,Strt} + g\;\underline{u}_{Dwn}^{B\,Strt} = 0 \\
\underline{a}_{End}^{B\,Strt} &= \underline{a}_{SF\,End}^{B\,Strt} + g\;\underline{u}_{Dwn}^{B\,Strt} = C_{B\,End}^{B\,Strt}\,\underline{a}_{SF\,End}^{B\,End} + g\;\underline{u}_{Dwn}^{B\,Strt} = 0
\end{aligned}
\tag{18}
$$

With (18), the equivalent truth version of (6) in Section 4.1 is

$$
\Delta\underline{a}^{B\,Strt} \equiv \underline{a}_{End}^{B\,Strt} - \underline{a}_{Strt}^{B\,Strt} = C_{B\,End}^{B\,Strt}\,\underline{a}_{SF\,End}^{B\,End} - \underline{a}_{SF\,Strt}^{B\,Strt} = 0 \tag{19}
$$

Then, from (19) and (18), the truth equivalent of (15) is

$$
\begin{gathered}
\Delta\underline{a}^{B\,Strt} = C_{B\,End}^{B\,Strt}\,\underline{a}_{SF\,End}^{B\,End} - \underline{a}_{SF\,Strt}^{B\,Strt} = 0 \qquad \Delta\underline{a}_{H}^{B\,Strt} \equiv \left(\Delta\underline{a}^{B\,Strt}\right)_{H} = 0 \\
a_{Strt\,Down}^{B\,Strt} = \underline{u}_{Dwn}^{B\,Strt} \cdot \underline{a}_{SF\,Strt}^{B\,Strt} + g = 0 \qquad a_{End\,Down}^{B\,Strt} = \underline{u}_{Dwn}^{B\,Strt} \cdot \left(C_{B\,End}^{B\,Strt}\,\underline{a}_{SF\,End}^{B\,End}\right) + g = 0 \\
\underline{u}_{Dwn}^{B\,Strt} = C_{NED}^{B\,Strt}\,\underline{u}_{Dwn}^{NED} \qquad \underline{u}_{Dwn}^{NED} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{T}
\end{gathered}
\tag{20}
$$

A truth model equivalent of (16) can also be identified for attitude determination

$$
\begin{gathered}
C_{B}^{B\,Strt} = I + \int_{t_{Strt}}^{t} \dot{C}_{B}^{B\,Strt}\,dt \\
\dot{C}_{B}^{B\,Strt} = C_{B}^{B\,Strt}\left(\underline{\omega}_{I:B}^{B}\times\right) - \left(\underline{\omega}_{I:E}^{B\,Strt}\times\right)C_{B}^{B\,Strt} \\
\underline{\omega}_{I:B}^{B\,Strt} = C_{NED}^{B\,Strt}\,\underline{\omega}_{I:E}^{NED} \qquad \underline{\omega}_{I:E}^{NED} = \begin{bmatrix} \omega_e \cos l & 0 & -\omega_e \sin l \end{bmatrix}^{T} \\
C_{B\,End}^{B\,Strt} = \left(C_{B}^{B\,Strt}\right)_{End} \quad @\, t = t_{End}
\end{gathered}
\tag{21}
$$

To develop (15) and (16) error model equivalents, we now equate the calculated (15) and (16) parameters (containing errors) to the (20) - (21) true parameters plus calculated parameter errors:

$$
\begin{aligned}
\hat{C}_{B\,End}^{B\,Strt} &= C_{B\,End}^{B\,Strt} + \delta\hat{C}_{B\,End}^{B\,Strt} \qquad & \hat{\underline{u}}_{Dwn}^{B\,Strt} &= \underline{u}_{Dwn}^{B\,Strt} + \delta\hat{\underline{u}}_{Dwn}^{B\,Strt} \\
\hat{\underline{a}}_{SF\,End}^{B\,End} &= \underline{a}_{SF\,End}^{B\,End} + \delta\hat{\underline{a}}_{SF\,End}^{B\,End} \qquad & \hat{\underline{a}}_{SF\,Strt}^{B\,Strt} &= \underline{a}_{SF\,Strt}^{B\,Strt} + \delta\hat{\underline{a}}_{SF\,Strt}^{B\,Strt}
\end{aligned}
\tag{22}
$$

where

$$
\delta\hat{C}_{B\,End}^{B\,Strt},\ \delta\hat{\underline{a}}_{SF\,End}^{B\,End},\ \delta\hat{\underline{a}}_{SF\,Strt}^{B\,Strt},\ \delta\hat{\underline{u}}_{Dwn}^{B\,Strt} \;=\; \text{Errors in } \hat{C}_{B\,End}^{B\,Strt},\ \hat{\underline{a}}_{SF\,End}^{B\,End},\ \hat{\underline{a}}_{SF\,Strt}^{B\,Strt},\hat{\underline{u}}_{Dwn}^{B\,Strt}.
$$

Applying (22) in (15) and with (20), substituting zero for $\Delta\underline{a}^{B\,Strt}$, $a_{Strt\,Down}^{B\,Strt}$, $a_{End\,Down}^{B\,Strt}$, then obtains

$$
\begin{aligned}
\Delta\underline{\hat{a}}^{B\,Strt} &= \hat{C}^{B\,Strt}_{B\,End}\ \underline{\hat{a}}^{B\,End}_{SF\,End} - \underline{\hat{a}}^{B\,Strt}_{SF\,Strt} \\
&= \left(C^{B\,Strt}_{B\,End} + \delta\hat{C}^{B\,Strt}_{B\,End}\right)\left(\underline{a}^{B\,End}_{SF\,End} + \delta\underline{\hat{a}}^{B\,End}_{SF\,End}\right) - \left(\underline{a}^{B\,Strt}_{SF\,Strt} + \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt}\right) \\
&= C^{B\,Strt}_{B\,End}\ \underline{a}^{B\,End}_{SF\,End} - \underline{a}^{B\,Strt}_{SF\,Strt} + \delta\hat{C}^{B\,Strt}_{B\,End}\ \underline{a}^{B\,End}_{SF\,End} \\
&\quad + C^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End} - \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt} + \delta\hat{C}^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End} \\
&= \delta\hat{C}^{B\,Strt}_{B\,End}\ C^{B\,End}_{B\,Strt}\ \underline{a}^{B\,Strt}_{SF\,End} + C^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End} - \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt} + \delta\hat{C}^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\hat{a}^{B\,Strt}_{Strt\,Dwn} &= \underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \underline{\hat{a}}^{B\,Strt}_{SF\,Strt} + g = \left(\underline{u}^{B\,Strt}_{Dwn} + \delta\underline{\hat{u}}^{B\,Strt}_{Dwn}\right) \cdot \left(\underline{a}^{B\,Strt}_{SF\,Strt} + \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt}\right) + g \\
&= \delta\underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \underline{a}^{B\,Strt}_{SF\,Strt} + \underline{u}^{B\,Strt}_{Dwn} \cdot \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt} + \delta\underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt}
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\hat{a}^{B\,Strt}_{End\,Down} &= \underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \left(\hat{C}^{B\,Strt}_{B\,End}\ \underline{\hat{a}}^{B\,End}_{SF\,End}\right) + g \\
&= \left(\underline{u}^{B\,Strt}_{Dwn} + \delta\underline{\hat{u}}^{B\,Strt}_{Dwn}\right) \cdot \left[\left(C^{B\,Strt}_{B\,End} + \delta\hat{C}^{B\,Strt}_{B\,End}\right)\left(\underline{a}^{B\,End}_{SF\,End} + \delta\underline{\hat{a}}^{B\,End}_{SF\,End}\right)\right] + g \\
&= \underline{u}^{B\,Strt}_{Dwn} \cdot \left(C^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End}\right) + \underline{u}^{B\,Strt}_{Dwn} \cdot \left(\delta\hat{C}^{B\,Strt}_{B\,End}\ \underline{a}^{B\,End}_{SF\,End}\right) + \delta\underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \left(C^{B\,Strt}_{B\,End}\ \underline{a}^{B\,End}_{SF\,End}\right) \\
&\quad + \underline{u}^{B\,Strt}_{Dwn} \cdot \left(\delta\hat{C}^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End}\right) + \delta\underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \left(C^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End} + \delta\hat{C}^{B\,Strt}_{B\,End}\ \underline{a}^{B\,End}_{SF\,End}\right) \\
&= \underline{u}^{B\,Strt}_{Dwn} \cdot \left(C^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End}\right) + \underline{u}^{B\,Strt}_{Dwn} \cdot \left(\delta\hat{C}^{B\,Strt}_{B\,End}\ C^{B\,End}_{B\,Strt}\ \underline{a}^{B\,Strt}_{SF\,End}\right) + \delta\underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \underline{a}^{B\,Strt}_{SF\,End} \\
&\quad + \underline{u}^{B\,Strt}_{Dwn} \cdot \left(\delta\hat{C}^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End}\right) + \delta\underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \left(C^{B\,Strt}_{B\,End}\ \delta\underline{\hat{a}}^{B\,End}_{SF\,End} + \delta\hat{C}^{B\,Strt}_{B\,End}\ \underline{a}^{B\,End}_{SF\,End}\right)
\end{aligned}
\tag{25}
$$

The remainder of this section develops analytical expressions for the $C^{B\,Strt}_{B\,End}$, $\underline{u}^{B\,Strt}_{Dwn}$, $\underline{a}^{B\,Strt}_{SF\,Strt}$, $\underline{a}^{B\,Strt}_{SF\,End}$, $\delta\underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \underline{a}^{B\,Strt}_{SF\,End}$, $\delta\underline{\hat{u}}^{B\,Strt}_{Dwn} \cdot \underline{a}^{B\,Strt}_{SF\,Strt}$, $\delta\hat{C}^{B\,Strt}_{B\,End}\,\delta\underline{\hat{a}}^{B\,End}_{SF\,End}$, and $\delta\hat{C}^{B\,Strt}_{B\,End}\ C^{B\,End}_{B\,Strt}\ \underline{a}^{B\,Strt}_{SF\,End}$ terms in (23) - (25); substitution in (23) - (25) for $\Delta\underline{\hat{a}}^{B\,Strt}_{H}$, $\hat{a}^{B\,Strt}_{Strt\,Dwn}$, and $\hat{a}^{B\,Strt}_{End\,Down}$; followed by a summary of results obtained. Included is a derivation of the rotation vector equivalent $\underline{\phi}^{B\,Strt}_{B\,End}$ for $\delta\hat{C}^{B\,Strt}_{B\,End}$ used in the final $\Delta\underline{\hat{a}}^{B\,Strt}_{H}$, $\hat{a}^{B\,Strt}_{Strt\,Dwn}$, $\hat{a}^{B\,Strt}_{End\,Down}$ error equations.

5.1 Developing The $C^{B\,Strt}_{B\,End}$ Term

For the $C^{B\,Strt}_{B\,End}$ term in (23) - (25):

$$C^{B_{Strt}}_{B_{End}} = C^{B_{Strt}}_{B^{Nom}_{Strt}} \, C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \, C^{B^{Nom}_{End}}_{B_{End}} \tag{26}$$

The $C^{B_{Strt}}_{B^{Nom}_{Strt}}$ and $C^{B^{Nom}_{End}}_{B_{End}}$ terms in (26) are nominally identity and can be expressed in terms of their rotation vector equivalent based on the rotation vector to direction cosine matrix conversion formula [2, Eq. (3.2.2.1-8)]

$$C^A_B = I + \frac{\sin\theta}{\theta}\left(\underline{\theta}^A \times\right) + \frac{1-\cos\theta}{\theta^2}\left(\underline{\theta}^A \times\right)^2 \tag{27}$$

where

C^A_B = Generalized direction cosine matrix that transforms vectors from general coordinate frame B to general coordinate frame A.

$\underline{\theta}^A$ = Rotation vector (in frame A coordinates) that would rotate frame A into frame B.

θ = Magnitude of $\underline{\theta}^A$.

Choosing B^{Nom}_{Strt} and B^{Nom}_{End} as the equivalent of frame A in (27), we write

$$\begin{aligned}
C^{B^{Nom}_{Strt}}_{B_{Strt}} &= I + \frac{\sin\alpha_{Strt}}{\alpha_{Strt}}\left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times\right) + \frac{1-\cos\alpha_{Strt}}{\alpha^2_{Strt}}\left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times\right)^2 \\
&= I + \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times\right) + \frac{1}{2}\left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times\right)^2 + \ldots \\
C^{B^{Nom}_{End}}_{B_{End}} &= I + \frac{\sin\alpha_{End}}{\alpha_{End}}\left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times\right) + \frac{1-\cos\alpha_{End}}{\alpha^2_{End}}\left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times\right)^2 \\
&= I + \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{End} \times\right) + \frac{1}{2}\left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times\right)^2 + \ldots
\end{aligned} \tag{28}$$

where

$\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt}$, $\underline{\alpha}^{B^{Nom}_{End}}_{End}$ = Rotation vector equivalent of $C^{B^{Nom}_{Strt}}_{B_{Strt}}$, $C^{B^{Nom}_{End}}_{B_{End}}$.

For small $\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt}$ and $\underline{\alpha}^{B^{Nom}_{Strt}}_{End}$ with $C^{B_{Strt}}_{B^{Nom}_{Strt}}$ identified as the transpose of $C^{B^{Nom}_{Strt}}_{B_{Strt}}$, (28) approximates as

$$C^{B_{Strt}}_{B^{Nom}_{Strt}} \approx I - \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \right) \qquad C^{B^{Nom}_{End}}_{B_{End}} \approx I + \left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times \right) \tag{29}$$

Substituting (29) into (26), assuming small $\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt}$, $\underline{\alpha}^{B^{Nom}_{Strt}}_{End}$, and dropping higher order terms then obtains for $C^{B_{Strt}}_{B_{End}}$ in (23) - (25):

$$\begin{aligned}
C^{B_{Strt}}_{B_{End}} &= \left[I - \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \right) \right] C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \left[I + \left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times \right) \right] \\
&= C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} - \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \right) C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} + C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times \right) - \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \right) \left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times \right) \\
&\approx C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} + C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times \right) - \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \right) C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \\
&= C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} + C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times \right) C^{B^{Nom}_{End}}_{B^{Nom}_{Strt}} C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} - \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \right) C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \qquad (30) \\
&= C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} + \left[\left(C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \underline{\alpha}^{B^{Nom}_{End}}_{End} \right) \times \right] C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} - \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \right) C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \\
&= C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} + \left[\left(C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \underline{\alpha}^{B^{Nom}_{End}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \right) \times \right] C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \\
&= C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} + \left[\left(\underline{\alpha}^{B^{Nom}_{Strt}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \right) \times \right] C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}}
\end{aligned}$$

5.2 Developing The $\underline{u}^{B_{Strt}}_{Dwn}$, $\underline{a}^{B_{Strt}}_{SF_{Strt}}$, And $\underline{a}^{B_{Strt}}_{SF_{End}}$ Terms

The $\underline{u}^{B_{Strt}}_{Dwn}$ term in (23) - (25) can be defined in terms of its equivalent in B^{Nom} coordinates as

$$\underline{u}^{B_{Strt}}_{Dwn} = C^{B_{Strt}}_{B^{Nom}_{Strt}} \, \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \tag{31}$$

or with (29) for $C^{B_{Strt}}_{B^{Nom}_{Strt}}$:

$$\underline{u}^{B_{Strt}}_{Dwn} = \left[I - \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \right) \right] \underline{u}^{B^{Nom}_{Strt}}_{Dwn} = \underline{u}^{B^{Nom}_{Strt}}_{Dwn} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \tag{32}$$

From (18), $\underline{a}_{SF_{Strt}}^{B_{Strt}}$ and $\underline{a}_{SF_{End}}^{B_{Strt}}$ in (23) - (25) are both $-g\,\underline{u}_{Dwn}^{B_{Strt}}$ so that with (32),

$$\underline{a}_{SF_{End}}^{B_{Strt}} = \underline{a}_{SF_{Strt}}^{B_{Strt}} = -\,g\left(\underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) \tag{33}$$

5.3 Developing The $\delta\hat{\underline{u}}_{Dwn}^{B_{Strt}}$, $\delta\hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \underline{a}_{SF_{End}}^{B_{Strt}}$ And $\delta\hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \underline{a}_{SF_{Strt}}^{B_{Strt}}$ Terms

The $\delta\hat{\underline{u}}_{Dwn}^{B_{Strt}}$ term in (24) - (25) can be defined based on the estimated value $\hat{\underline{u}}_{Dwn}^{B_{Strt}}$ being along the nominal vertical, i.e., along $\underline{u}_{Dwn}^{B_{Strt}^{Nom}}$. From (32):

$$\delta\hat{\underline{u}}_{Dwn}^{B_{Strt}} \equiv \hat{\underline{u}}_{Dwn}^{B_{Strt}} - \underline{u}_{Dwn}^{B_{Strt}} = \underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \left(\underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) = \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}} \tag{34}$$

With (34) and (33), the $\delta\hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \underline{a}_{SF_{End}}^{B_{Strt}}$ and $\delta\hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \underline{a}_{SF_{Strt}}^{B_{Strt}}$ terms in (24) - (25) then become:

$$\begin{aligned}\delta\hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \underline{a}_{SF_{End}}^{B_{Strt}} = \delta\hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \underline{a}_{SF_{Strt}}^{B_{Strt}} &= -g\left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) \cdot \left(\underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) \\ &= g\left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) \cdot \left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right)\end{aligned} \tag{35}$$

5.4 Developing The $\delta\hat{C}_{B_{End}}^{B_{Strt}}\,\delta\hat{\underline{a}}_{SF_{End}}^{B_{End}}$ And $\delta\hat{C}_{B_{End}}^{B_{Strt}}\,C_{B_{Strt}}^{B_{End}}\,\underline{a}_{SF_{End}}^{B_{Strt}}$ Terms

The $\delta\hat{C}_{B_{End}}^{B_{Strt}}$ term in (23) and (25) can be defined in terms of a small angle rotation error vector equivalent. First recognize from (16) and (21) that at the start of the rotation sequence, $\hat{C}_{B}^{B_{Strt}}$ and $C_{B}^{B_{Strt}}$ are both identity. Then we can write

$$\hat{C}_{B}^{B_{Strt}} = C_{B}^{B_{Strt}}\,C_{\hat{B}}^{B} \tag{36}$$

where

$\hat{B} =$ Implicit B frame imbedded in $C_{\hat{B}}^{B}$ that differs from the true B frame because of errors induced in the (16) integration process for $\hat{C}_{B}^{B_{Strt}}$ compared with the (21) true integration process for $C_B^{B\,Strt}$. Since both $\hat{C}_B^{B\,Strt}$ and $C_B^{B\,Strt}$ are initialized at identity at the start of the rotation sequence, their difference at the end of the sequence is due to errors in the (16) process $\dot{\hat{C}}_B^{B\,Strt}$ integrand.

Following the Section 4.2.1 development for $C_{B_{Strt}^{Nom}}^{B\,Strt}$ in (29), we define $C_{\hat{B}}^{B}$ in terms of a small rotation vector error as

$$C_{\hat{B}}^{B} \approx I + \left(\underline{\phi}^B \times\right) + \frac{1}{2}\left(\underline{\phi}^B \times\right)^2 \tag{37}$$

But by definition,

$$\delta\hat{C}_B^{B\,Strt} \equiv \hat{C}_B^{B\,Strt} - C_B^{B\,Strt} \tag{38}$$

Substituting (36) and (37) in (38) then finds for $\delta\hat{C}_B^{B\,Strt}$:

$$\begin{aligned}
\delta\hat{C}_B^{B\,Strt} &= C_B^{B\,Strt}\, C_{\hat{B}}^{B} - C_B^{B\,Strt} = C_B^{B\,Strt}\left(C_{\hat{B}}^{B} - I\right) \approx C_B^{B\,Strt}\left(\underline{\phi}^B \times\right)\left[I + \frac{1}{2}\left(\underline{\phi}^B \times\right)\right] \\
&= C_B^{B\,Strt}\left(\underline{\phi}^B \times\right) C_{B_{Strt}}^{B}\, C_B^{B\,Strt}\left[I + \frac{1}{2}\left(\underline{\phi}^B \times\right)\right] = \left(\underline{\phi}^{B\,Strt} \times\right) C_B^{B\,Strt}\left[I + \frac{1}{2}\left(\underline{\phi}^B \times\right)\right] \\
&= \left[\left(\underline{\phi}^{B\,Strt} \times\right) + \frac{1}{2}\left(\underline{\phi}^{B\,Strt} \times\right)^2\right] C_B^{B\,Strt}
\end{aligned} \tag{39}$$

or at the end of the rotation sequence,

$$\delta\hat{C}_{B\,End}^{B\,Strt} = \left[\left(\underline{\phi}_{End}^{B\,Strt} \times\right) + \frac{1}{2}\left(\underline{\phi}_{End}^{B\,Strt} \times\right)^2\right] C_{B\,End}^{B\,Strt} \tag{40}$$

From (40) and neglecting third order terms, $\delta\hat{C}_{B\,End}^{B\,Strt}\, \delta\hat{\underline{a}}_{SF\,End}^{B\,End}$ in (23) then approximates as

$$\delta\hat{C}_{B\,End}^{B\,Strt}\, \delta\hat{\underline{a}}_{SF\,End}^{B\,End} \approx \underline{\phi}_{End}^{B\,Strt} \times \left(C_{B\,End}^{B\,Strt}\, \delta\hat{\underline{a}}_{SF\,End}^{B\,End}\right) \approx \underline{\phi}_{End}^{B\,Strt} \times \left(C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, \delta\hat{\underline{a}}_{SF\,End}^{B\,End}\right) \tag{41}$$

From (40) with $\underline{a}_{SF\,End}^{B\,Strt}$ from (33), the $\delta\hat{C}_{B\,End}^{B\,Strt}\, C_{B\,Strt}^{B\,End}\, \underline{a}_{SF\,End}^{B\,Strt}$ term in (23) becomes

$$
\begin{aligned}
\delta\hat{C}_{B\,End}^{B\,Strt}\, C_{B\,Strt}^{B\,End}\, \underline{a}_{SF\,End}^{B\,Strt} &= \left[\left(\underline{\phi}_{End}^{B\,Strt}\times\right)+\frac{1}{2}\left(\underline{\phi}_{End}^{B\,Strt}\times\right)^2\right] C_{B\,End}^{B\,Strt}\, C_{B\,Strt}^{B\,End}\, \underline{a}_{SF\,End}^{B\,Strt} \\
&= \left[\left(\underline{\phi}_{End}^{B\,Strt}\times\right)+\frac{1}{2}\left(\underline{\phi}_{End}^{B\,Strt}\times\right)^2\right]\times \underline{a}_{SF\,End}^{B\,Strt} \\
&= -\,g\left[\left(\underline{\phi}_{End}^{B\,Strt}\times\right)+\frac{1}{2}\left(\underline{\phi}_{End}^{B\,Strt}\times\right)^2\right]\left(\underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) \\
&\approx -g\left\{\underline{\phi}_{End}^{B\,Strt}\times\left[\underline{u}_{Dwn}^{B_{Strt}^{Nom}} + \left(\frac{1}{2}\underline{\phi}_{End}^{B\,Strt} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right)\times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right]\right\}
\end{aligned}
\tag{42}
$$

5.5 ANALYTICAL EQUIVALENTS OF THE $\hat{a}_{Strt\,Dwn}^{B\,Strt}$, $\hat{a}_{End\,Dwn}^{B\,Strt}$, $\Delta\hat{\underline{a}}_{H}^{B\,Strt}$ MEASUREMENTS

Applying the Section 5.1 - 5.4 results to Eqs. (23) - (25) and dropping third order terms obtains the second order equivalents to $\hat{a}_{Strt\,Dwn}^{B\,Strt}$, $\hat{a}_{End\,Dwn}^{B\,Strt}$, and $\Delta\hat{\underline{a}}_{H}^{B\,Strt}$.

For the $\hat{a}_{Strt\,Dwn}^{B\,Strt}$ and $\hat{a}_{End\,Dwn}^{B\,Strt}$ measurements we substitute $C_{B\,End}^{B\,Strt}$ from (30), $\underline{u}_{Dwn}^{B\,Strt}$ from (32), $\underline{a}_{SF\,Strt}^{B\,Strt}$ and $\underline{a}_{SF\,End}^{B\,Strt}$ from (33), $\delta\hat{\underline{u}}_{Dwn}^{B\,Strt}$ from (34), $\delta\hat{\underline{u}}_{Dwn}^{B\,Strt}\cdot\underline{a}_{SF\,End}^{B\,Strt}$ and $\delta\hat{\underline{u}}_{Dwn}^{B\,Strt}\cdot\underline{a}_{SF\,Strt}^{B\,Strt}$ from (35), $\delta\hat{C}_{B\,End}^{B\,Strt}$ from (40), $\delta\hat{C}_{B\,End}^{B\,Strt}\,\delta\hat{\underline{a}}_{SF\,End}^{B\,End}$ from (41), and $\delta\hat{C}_{B\,End}^{B\,Strt}\, C_{B\,Strt}^{B\,End}\,\underline{a}_{SF\,End}^{B\,Strt}$ from (42) into (24) - (25) to obtain the second order equivalents to $\hat{a}_{Strt\,Dwn}^{B\,Strt}$ and $\hat{a}_{End\,Dwn}^{B\,Strt}$ in (15):

$$
\begin{aligned}
\hat{a}_{Strt\,Dwn}^{B\,Strt} &= \delta\hat{\underline{u}}_{Dwn}^{B\,Strt}\cdot\underline{a}_{SF\,Strt}^{B\,Strt} + \underline{u}_{Dwn}^{B\,Strt}\cdot\delta\hat{\underline{a}}_{SF\,Strt}^{B\,Strt} + \delta\hat{\underline{u}}_{Dwn}^{B\,Strt}\cdot\delta\hat{\underline{a}}_{SF\,Strt}^{B\,Strt} \\
&= g\left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\times\underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right)\cdot\left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\times\underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) + \left(\underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\times\underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right)\cdot\delta\hat{\underline{a}}_{SF\,Strt}^{B\,Strt} \\
&\qquad + \left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\times\underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right)\cdot\delta\hat{\underline{a}}_{SF\,Strt}^{B\,Strt} \\
&= \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\cdot\delta\hat{\underline{a}}_{SF\,Strt}^{B\,Strt} + g\left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\times\underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right)\cdot\left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\times\underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right)
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
\hat{a}^{B_{Strt}}_{End\,Dwn} &= \underline{u}^{B_{Strt}}_{Dwn} \cdot \left(C^{B_{Strt}}_{B_{End}} \, \delta\hat{\underline{a}}^{B_{End}}_{SF\,End} \right) + \underline{u}^{B_{Strt}}_{Dwn} \cdot \left(\delta\hat{C}^{B_{Strt}}_{B_{End}} \, C^{B_{End}}_{B_{Strt}} \, \underline{a}^{B_{Strt}}_{SF\,End} \right) + \delta\hat{\underline{u}}^{B_{Strt}}_{Dwn} \cdot \underline{a}^{B_{Strt}}_{SF\,End} \\
&\quad + \underline{u}^{B_{Strt}}_{Dwn} \cdot \left(\delta\hat{C}^{B_{Strt}}_{B_{End}} \, \delta\hat{\underline{a}}^{B_{End}}_{SF\,End} \right) + \delta\hat{\underline{u}}^{B_{Strt}}_{Dwn} \cdot \left(C^{B_{Strt}}_{B_{End}} \, \delta\hat{\underline{a}}^{B_{End}}_{SF\,End} + \delta\hat{C}^{B_{Strt}}_{B_{End}} \, \underline{a}^{B_{End}}_{SF\,End} \right) \\
&= \left(\underline{u}^{B^{Nom}_{Strt}}_{Dwn} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \right) \cdot \left\langle \left\{ C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} + \left[\left(\underline{\alpha}^{B^{Nom}_{Strt}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \right) \times \right] C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \right\} \delta\hat{\underline{a}}^{B_{End}}_{SF\,End} \right\rangle \\
&\quad - g \left(\underline{u}^{B^{Nom}_{Strt}}_{Dwn} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \right) \cdot \left\{ \underline{\phi}^{B_{Strt}}_{End} \times \left[\underline{u}^{B^{Nom}_{Strt}}_{Dwn} + \left(\frac{1}{2} \underline{\phi}^{B_{Strt}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \right) \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \right] \right\} \\
&\qquad - g \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \right) \cdot \left(\underline{u}^{B^{Nom}_{Strt}}_{Dwn} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \right) \\
&\qquad + \left(\underline{u}^{B^{Nom}_{Strt}}_{Dwn} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \right) \cdot \left[\underline{\phi}^{B_{Strt}}_{End} \times \left(C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \, \delta\hat{\underline{a}}^{B_{End}}_{SF\,End} \right) \right] \qquad (44) \\
&\qquad + \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \right) \cdot \left\langle \begin{array}{l} \left\{ C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} + \left[\left(\underline{\alpha}^{B^{Nom}_{Strt}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \right) \times \right] C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \right\} \delta\hat{\underline{a}}^{B_{End}}_{SF\,End} \\ \quad + \left[\left(\underline{\phi}^{B_{Strt}}_{End} \times \right) + \frac{1}{2} \left(\underline{\phi}^{B_{Strt}}_{End} \times \right)^2 \right] C^{B_{Strt}}_{B_{End}} \, \underline{a}^{B_{End}}_{SF\,End} \end{array} \right\rangle \\
&\approx \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \cdot \left(C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \, \delta\hat{\underline{a}}^{B_{End}}_{SF\,End} \right) + \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \cdot \left\{ \left[\left(\underline{\phi}^{B_{Strt}}_{End} + \underline{\alpha}^{B^{Nom}_{Strt}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \right) \times \right] \left(C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \, \delta\hat{\underline{a}}^{B_{End}}_{SF\,End} \right) \right\} \\
&\qquad - g \, \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \cdot \left\{ \underline{\phi}^{B_{Strt}}_{End} \times \left[\left(\frac{1}{2} \underline{\phi}^{B_{Strt}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \right) \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \right] \right\} \\
&\qquad + \left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \right) \cdot \left[g \left(\underline{\phi}^{B_{Strt}}_{End} + \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \right) \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn} + \underline{\phi}^{B_{Strt}}_{End} \times \underline{a}^{B_{Strt}}_{SF\,End} \right]
\end{aligned}
$$

For the $\Delta\hat{\underline{a}}^{B_{Strt}}$ measurement we substitute $C^{B_{Strt}}_{B_{End}}$ from (30), $\delta\hat{C}^{B_{Strt}}_{B_{End}} \, \delta\hat{\underline{a}}^{B_{End}}_{SF\,End}$ from (41), and $\delta\hat{C}^{B_{Strt}}_{B_{End}} \, C^{B_{End}}_{B_{Strt}} \, \underline{a}^{B_{Strt}}_{SF\,End}$ from (42) into (23) to obtain the second order model equivalent to $\Delta\hat{\underline{a}}^{B_{Strt}}$ in (15):

$$
\begin{aligned}
\Delta \underline{\hat{a}}^{B\,Strt} &= \delta\hat{C}^{B\,Strt}_{B\,End}\, C^{B\,End}_{B\,Strt}\, \underline{a}^{B\,Strt}_{SF\,End} + C^{B\,Strt}_{B\,End}\, \delta\underline{\hat{a}}^{B\,End}_{SF\,End} - \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt} + \delta\hat{C}^{B\,Strt}_{B\,End}\, \delta\underline{\hat{a}}^{B\,End}_{SF\,End} \\
&= -g\left\{\underline{\phi}^{B\,Strt}_{End} \times \left[\left(\underline{u}^{B^{Nom}_{Strt}}_{Dwn} + \left(\frac{1}{2}\underline{\phi}^{B\,Strt}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt}\right)\times \underline{u}^{B^{Nom}_{Strt}}_{Dwn}\right)\right]\right\} \\
&\quad + \left\{C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} + \left[\left(\underline{\alpha}^{B^{Nom}_{Strt}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt}\right)\times\right] C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}}\right\} \delta\underline{\hat{a}}^{B\,End}_{SF\,End} - \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt} \\
&\quad + \underline{\phi}^{B\,Strt}_{End} \times \left(C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}}\, \delta\underline{\hat{a}}^{B\,End}_{SF\,End}\right) \\
&= g\, \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \times \underline{\phi}^{B\,Strt}_{End} + C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}}\, \delta\underline{\hat{a}}^{B\,End}_{SF\,End} - \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt} \\
&\quad - g\, \underline{\phi}^{B\,Strt}_{End} \times \left[\left(\frac{1}{2}\underline{\phi}^{B\,Strt}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt}\right)\times \underline{u}^{B^{Nom}_{Strt}}_{Dwn}\right] + \left[\left(\underline{\alpha}^{B^{Nom}_{Strt}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt}\right)\times\right] C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}}\, \delta\underline{\hat{a}}^{B\,End}_{SF\,End} \\
&\quad + \underline{\phi}^{B\,Strt}_{End} \times \left(C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}}\, \delta\underline{\hat{a}}^{B\,End}_{SF\,End}\right) \\
&= g\left[\underline{u}^{B^{Nom}_{Strt}}_{Dwn} + \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \times \left(\frac{1}{2}\underline{\phi}^{B\,Strt}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt}\right)\right] \times \underline{\phi}^{B\,Strt}_{End} + C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}}\, \delta\underline{\hat{a}}^{B\,End}_{SF\,End} - \delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt} \\
&\quad + \left[\left(\underline{\phi}^{B\,Strt}_{End} + \underline{\alpha}^{B^{Nom}_{Strt}}_{End} - \underline{\alpha}^{B^{Nom}_{Strt}}_{Strt}\right)\times\right] C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}}\, \delta\underline{\hat{a}}^{B\,End}_{SF\,End}
\end{aligned}
\tag{45}
$$

The $\delta\underline{\hat{a}}^{B\,End}_{SF\,End}$ and $\delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt}$ terms in (43) - (45) are to second order accuracy from (A-22) and (A-24) in Appendix A:

$$
\begin{aligned}
\delta\underline{\hat{a}}^{B\,Strt}_{SF\,Strt} &= -\,g\left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym}\, A^{B\,Strt}_{SFSign}\right)\underline{u}^{B^{Nom}_{Strt}}_{Dwn} + \underline{\lambda}_{Bias} \\
&+ g\left(\lambda_{LinScal/Mis} + \lambda_{Asym}\, A^{B\,End}_{SFSign}\right)\left(\underline{\alpha}^{B^{Nom}_{Strt}}_{Strt} \times \underline{u}^{B^{Nom}_{Strt}}_{Dwn}\right) + \underline{\lambda}_{Quant\,Strt} + \underline{\lambda}_{Rndm\,Strt}
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
\underline{\hat{a}}^{B\,End}_{SF\,End} &= -\,g\left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym}\, A^{B\,End}_{SFSign}\right)\underline{u}^{B^{Nom}_{End}}_{Dwn} + \underline{\lambda}_{Bias} \\
&+ g\left(\lambda_{LinScal/Mis} + \lambda_{Asym}\, A^{B\,End}_{SFSign}\right)\left(\underline{\alpha}^{B^{Nom}_{End}}_{End} \times \underline{u}^{B^{Nom}_{End}}_{Dwn}\right) + \underline{\lambda}_{Quant\,End} + \underline{\lambda}_{Rndm\,End}
\end{aligned}
\tag{47}
$$

Equations (45) – (47) define a second order accuracy analytical equivalent to the (15) measurements as functions of sensor calibration error, IMU mounting error, and (16) attitude computation error caused by initialization uncertainty and rotation fixture error in rotation sequence execution. It remains to derive an equation for attitude error $\underline{\phi}^{B\,Strt}_{End}$ in (45).

5.6 MODELING THE $\underline{\phi}_{End}^{B\,Strt}$ ATTITUDE ERROR

This section derives an analytical model for the $\underline{\phi}_{End}^{B\,Strt}$ rotation vector in (43), the equivalent to the $\delta\hat{C}_{B\,End}^{B\,Strt}$ error in $\hat{C}_{B\,End}^{B\,Strt}$ computed attitude. The $\delta\hat{C}_{B\,End}^{B\,Strt}$ error is the result of generating $\hat{C}_{B}^{B\,Strt}$ in (16) during the rotation sequence as an integration process with imperfect gyro input data ($\hat{\underline{\omega}}_{I:B}^{B}$), uncertainty in the B_{Strt} frame earth rate components ($\hat{\underline{\omega}}_{I:E}^{B\,Strt}$), and rotation fixture inaccuracy in executing SRT rotations.

We first develop an equivalent for $\delta\hat{C}_{B\,End}^{B\,Strt}$ as a function of the integrals of $\dot{\hat{C}}_{B}^{B\,Strt}$ and $\dot{C}_{B}^{B\,Strt}$ over the (16) and (21) rotation sequence. Defining $\delta\hat{C}_{B\,End}^{B\,Strt}$ as $\delta\hat{C}_{B}^{B\,Strt}$ for arbitrary B at the end of a rotation sequence:

$$\delta\hat{C}_{B}^{B\,Strt} \equiv \hat{C}_{B}^{B\,Strt} - C_{B}^{B\,Strt} \qquad \delta\hat{C}_{B\,End}^{B\,Strt} = \delta\hat{C}_{B}^{B\,Strt} \;\; @ \;\; t = t_{End} \tag{48}$$

From (16) and (21), $\delta\hat{C}_{B}^{B\,Strt}$ in (48) is

$$\begin{aligned} \delta\hat{C}_{B}^{B\,Strt} &\equiv \hat{C}_{B}^{B\,Strt} - C_{B}^{B\,Strt} = \int_{t_{Strt}}^{t} \dot{\hat{C}}_{B}^{B\,Strt} dt - \int_{t_{Strt}}^{t} \dot{C}_{B}^{B\,Strt} dt \\ &= \int_{t_{Strt}}^{t} \left(\dot{\hat{C}}_{B}^{B\,Strt} - \dot{C}_{B}^{B\,Strt} \right) dt = \int_{t_{Strt}}^{t} \delta\dot{\hat{C}}_{B}^{B\,Strt} dt \end{aligned} \tag{49}$$

for which

$$\delta\dot{\hat{C}}_{B}^{B\,Strt} \equiv \dot{\hat{C}}_{B}^{B\,Strt} - \dot{C}_{B}^{B\,Strt} \tag{50}$$

Substituting $\dot{\hat{C}}_{B}^{B\,Strt}$ from (16) and $\dot{C}_{B}^{B\,Strt}$ from (21) in (50) then obtains

$$\delta\dot{\hat{C}}_{B}^{B\,Strt} = \hat{C}_{B}^{B\,Strt}\left(\hat{\underline{\omega}}_{I:B}^{B}\times\right) - \left(\hat{\underline{\omega}}_{I:E}^{B\,Strt}\times\right)\hat{C}_{B}^{B\,Strt} - C_{B}^{B\,Strt}\left(\underline{\omega}_{I:B}^{B}\times\right) + \left(\underline{\omega}_{I:E}^{B\,Strt}\times\right)C_{B}^{B\,Strt} \tag{51}$$

Now define

$$\hat{\underline{\omega}}_{I:B}^{B} = \underline{\omega}_{I:B}^{B} + \delta\hat{\underline{\omega}}_{I:B}^{B} \qquad \hat{\underline{\omega}}_{I:E}^{B\,Strt} = \underline{\omega}_{I:E}^{B\,Strt} + \delta\hat{\underline{\omega}}_{I:E}^{B\,Strt} \qquad \hat{C}_{B}^{B\,Strt} = C_{B}^{B\,Strt} + \delta\hat{C}_{B}^{B\,Strt} \tag{52}$$

Substituting (52) in (51) gives

$$
\begin{aligned}
\delta\dot{\hat{C}}_B^{B\,Strt} &= \hat{C}_B^{B\,Strt}\left(\underline{\hat{\omega}}_{I:B}^{B}\times\right)-\left(\underline{\hat{\omega}}_{I:E}^{B\,Strt}\times\right)\hat{C}_B^{B\,Strt} - C_B^{B\,Strt}\left(\underline{\omega}_{I:B}^{B}\times\right)+\left(\underline{\omega}_{I:E}^{B\,Strt}\times\right)C_B^{B\,Strt} \\
&=\left(C_B^{B\,Strt} + \delta\hat{C}_B^{B\,Strt}\right)\left[\left(\underline{\omega}_{I:B}^{B} + \delta\underline{\hat{\omega}}_{I:B}^{B}\right)\times\right]-\left[\left(\underline{\omega}_{I:E}^{B\,Strt} + \delta\underline{\hat{\omega}}_{I:E}^{B\,Strt}\right)\times\right]\left(C_B^{B\,Strt} + \delta\hat{C}_B^{B\,Strt}\right) \\
&\qquad - C_B^{B\,Strt}\left(\underline{\omega}_{I:B}^{B}\times\right)+\left(\underline{\omega}_{I:E}^{B\,Strt}\times\right)C_B^{B\,Strt} \\
&= C_B^{B\,Strt}\left(\delta\underline{\hat{\omega}}_{I:B}^{B}\times\right)-\delta\underline{\hat{\omega}}_{I:E}^{B\,Strt}\times C_B^{B\,Strt}+\delta\hat{C}_B^{B\,Strt}\left(\underline{\omega}_{I:B}^{B}\times\right)-\left(\underline{\omega}_{I:E}^{B\,Strt}\times\right)\delta\hat{C}_B^{B\,Strt} \\
&\qquad +\delta\hat{C}_B^{B\,Strt}\left(\delta\underline{\omega}_{I:B}^{B}\times\right)-\delta\underline{\hat{\omega}}_{I:E}^{B\,Strt}\times\delta\hat{C}_B^{B\,Strt} \\
&\approx C_B^{B\,Strt}\left(\delta\underline{\omega}_{I:B}^{B}\times\right)-\delta\underline{\hat{\omega}}_{I:E}^{B\,Strt}\times C_B^{B\,Strt}+\delta\hat{C}_B^{B\,Strt}\left(\underline{\omega}_{I:B}^{B}\times\right)-\left(\underline{\omega}_{I:E}^{B\,Strt}\times\right)\delta\hat{C}_B^{B\,Strt}
\end{aligned}
\tag{53}
$$

As in (40), we incorporate

$$
\delta\hat{C}_B^{B\,Strt}=\left(\underline{\phi}^{B\,Strt}\times\right)C_B^{B\,Strt}
\tag{54}
$$

Substituting (54) in (53) finds

$$
\begin{aligned}
\delta\dot{\hat{C}}_B^{B\,Strt} &\approx C_B^{B\,Strt}\left(\delta\underline{\hat{\omega}}_{I:B}^{B}\times\right)-\delta\underline{\hat{\omega}}_{I:E}^{B\,Strt}\times C_B^{B\,Strt} \\
&+\left(\underline{\phi}^{B\,Strt}\times\right)C_B^{B\,Strt}\left(\underline{\omega}_{I:B}^{B}\times\right)-\left(\underline{\omega}_{I:E}^{B\,Strt}\times\right)\left(\underline{\phi}^{B\,Strt}\times\right)C_B^{B\,Strt}
\end{aligned}
\tag{55}
$$

But from (54) with (21):

$$
\begin{aligned}
\delta\dot{\hat{C}}_B^{B\,Strt} &=\left(\dot{\underline{\phi}}^{B\,Strt}\times\right)C_B^{B\,Strt}+\left(\underline{\phi}^{B\,Strt}\times\right)\dot{C}_B^{B\,Strt} \\
&=\left(\dot{\underline{\phi}}^{B\,Strt}\times\right)C_B^{B\,Strt}+\left(\underline{\phi}^{B\,Strt}\times\right)\left[C_B^{B\,Strt}\left(\underline{\omega}_{I:B}^{B}\times\right)-\left(\underline{\omega}_{I:E}^{B\,Strt}\times\right)C_B^{B\,Strt}\right]
\end{aligned}
\tag{56}
$$

Equating (56) to (55) obtains with rearrangement

$$\left(\dot{\underline{\phi}}^{B_{Strt}} \times\right) C_B^{B_{Strt}} = C_B^{B_{Strt}} \left(\delta \widehat{\underline{\omega}}_{I:B}^{B} \times\right) - \left(\delta \widehat{\underline{\omega}}_{I:E}^{B_{Strt}} \times\right) C_B^{B_{Strt}}$$
$$-\left(\underline{\phi}^{B_{Strt}} \times\right)\left[C_B^{B_{Strt}} \left(\underline{\omega}_{I:B}^{B} \times\right) - \left(\underline{\omega}_{I:E}^{B_{Strt}} \times\right) C_B^{B_{Strt}} \right]$$
$$+\left(\underline{\phi}^{B_{Strt}} \times\right) C_B^{B_{Strt}} \left(\underline{\omega}_{I:B}^{B} \times\right) - \left(\underline{\omega}_{I:E}^{B_{Strt}} \times\right)\left(\underline{\phi}^{B_{Strt}} \times\right) C_B^{B_{Strt}}$$
$$= C_B^{B_{Strt}} \left(\delta \widehat{\underline{\omega}}_{I:B}^{B} \times\right) C_{B_{Strt}}^{B}\, C_B^{B_{Strt}} - \left(\delta \widehat{\underline{\omega}}_{I:E}^{B_{Strt}} \times\right) C_B^{B_{Strt}} \tag{57}$$
$$+\left(\underline{\phi}^{B_{Strt}} \times\right)\left(\underline{\omega}_{IE}^{B_{Strt}} \times\right) C_B^{B_{Strt}} - \left(\underline{\omega}_{I:E}^{B_{Strt}} \times\right)\left(\underline{\phi}^{B_{Strt}} \times\right) C_B^{B_{Strt}}$$
$$= \left[\left(C_B^{B_{Strt}} \delta \widehat{\underline{\omega}}_{I:B}^{B}\right) \times\right] C_B^{B_{Strt}} - \left(\delta \widehat{\underline{\omega}}_{I:E}^{B_{Strt}} \times\right) C_B^{B_{Strt}} + \left[\left(\underline{\phi}^{B_{Strt}} \times \underline{\omega}_{I:E}^{B_{Strt}}\right) \times\right] C_B^{B_{Strt}}$$

Thus,

$$\dot{\underline{\phi}}^{B_{Strt}} = C_B^{B_{Strt}} \delta \widehat{\underline{\omega}}_{I:B}^{B} - \delta \widehat{\underline{\omega}}_{I:E}^{B_{Strt}} + \underline{\phi}^{B_{Strt}} \times \underline{\omega}_{I:E}^{B_{Strt}} \tag{58}$$

The $C_B^{B_{Strt}}$ term in (58) derives from (30) with arbitrary B substituted for B_{End} :

$$C_B^{B_{Strt}} = C_{B^{Nom}}^{B_{Strt}^{Nom}} + \left[\left(\underline{\alpha}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times\right] C_{B^{Nom}}^{B_{Strt}^{Nom}} \tag{59}$$

With (59) and (B-23) to second order accuracy for $\delta \widehat{\underline{\omega}}_{I:B}^{B}$, the $C_B^{B_{Strt}} \delta \widehat{\underline{\omega}}_{I:B}^{B}$ term in (58) becomes with rearrangement

$$\begin{aligned}
C_B^{B_{Strt}} \delta \widehat{\underline{\omega}}_{I:B}^{B} &= C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \Omega_{EBSign}^{B}\right)\left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} + \underline{\omega}_{I:E}^{B^{Nom}}\right) \\
&+ \left(\underline{\alpha}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times \left[C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \Omega_{EBSign}^{B}\right) \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right] \\
&+ C_{B^{Nom}}^{B_{Strt}^{Nom}} \left[\left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \Omega_{EBSign}^{B}\right)\left(\dot{\underline{\alpha}}^{B^{Nom}} - \underline{\alpha}^{B^{Nom}} \times \underline{\omega}_{E:B^{Nom}}^{B^{Nom}}\right)\right] \\
&+ C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\underline{\kappa}_{Bias} + \delta \underline{\omega}_{Quant} + \delta \underline{\omega}_{Rand}\right)
\end{aligned} \tag{60}$$

The $\delta \widehat{\underline{\omega}}_{I:E}^{B_{Strt}}$ term in (58) is based on $\delta \widehat{\underline{\omega}}_{I:E}^{B_{Strt}} \equiv \widehat{\underline{\omega}}_{I:E}^{B_{Strt}} - \underline{\omega}_{I:E}^{B_{Strt}}$. Using (29), $\underline{\omega}_{I:E}^{B_{Strt}}$ is given by

$$\underline{\omega}_{I:E}^{B_{Strt}} = C_{B_{Strt}^{Nom}}^{B_{Strt}} \, \underline{\omega}_{I:E}^{B_{Strt}^{Nom}} = \left[I - \left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \right) \right] \underline{\omega}_{I:E}^{B_{Strt}^{Nom}} = \underline{\omega}_{I:E}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{\omega}_{I:E}^{B_{Strt}^{Nom}} \quad (61)$$

The estimated earth rate $\hat{\underline{\omega}}_{I:E}^{B_{Strt}}$ expression in (16) approximates $\hat{\underline{\omega}}_{I:E}^{B_{Strt}} = \underline{\omega}_{I:E}^{B_{Strt}^{Nom}}$. Then with (61) for $\underline{\omega}_{I:E}^{B_{Strt}}$,

$$\delta \hat{\underline{\omega}}_{I:E}^{B_{Strt}} \equiv \hat{\underline{\omega}}_{I:E}^{B_{Strt}} - \underline{\omega}_{I:E}^{B_{Strt}} = \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{\omega}_{I:E}^{B_{Strt}^{Nom}} \quad (62)$$

With (60) and (62), (58) becomes

$$\begin{aligned}
\dot{\underline{\phi}}^{B_{Strt}} = {} & C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym} \, \Omega_{EBSign}^{B} \right) \left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} + \underline{\omega}_{I:E}^{B^{Nom}} \right) \\
& + \left(\underline{\alpha}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \right) \times \left[C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym} \, \Omega_{EBSign}^{B} \right) \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right] \\
& + C_{B^{Nom}}^{B_{Strt}^{Nom}} \left[\left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym} \, \Omega_{EBSign}^{B} \right) \left(\dot{\underline{\alpha}}^{B^{Nom}} - \underline{\alpha}^{B^{Nom}} \times \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right) \right] \\
& + C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\underline{\kappa}_{Bias} + \delta \underline{\omega}_{Quant} + \delta \underline{\omega}_{Rand} \right) - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{\omega}_{I:E}^{B_{Strt}^{Nom}} + \underline{\phi}^{B_{Strt}} \times \underline{\omega}_{I:E}^{B_{Strt}}
\end{aligned} \quad (63)$$

Integrating (63) over the rotation sequence yields $\underline{\phi}_{End}^{B_{Strt}}$ to second order accuracy for Eqs. (43) - (45):

$$\underline{\phi}^{B_{Strt}} = \int_{t_{Strt}}^{t} \dot{\underline{\phi}}^{B_{Strt}} \, dt$$

$$\begin{aligned}
\dot{\underline{\phi}}^{B_{Strt}} = {} & C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym} \, \Omega_{EBSign}^{B} \right) \left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} + \underline{\omega}_{I:E}^{B^{Nom}} \right) \\
& + \left(\underline{\alpha}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \right) \times \left[C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym} \, \Omega_{EBSign}^{B} \right) \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right] \\
& + C_{B^{Nom}}^{B_{Strt}^{Nom}} \left[\left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym} \, \Omega_{EBSign}^{B} \right) \left(\dot{\underline{\alpha}}^{B^{Nom}} - \underline{\alpha}^{B^{Nom}} \times \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right) \right] \\
& + C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\underline{\kappa}_{Bias} + \delta \underline{\omega}_{Quant} + \delta \underline{\omega}_{Rand} \right) - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{\omega}_{I:E}^{B_{Strt}^{Nom}} + \underline{\phi}^{B_{Strt}} \times \underline{\omega}_{I:E}^{B_{Strt}}
\end{aligned} \quad (64)$$

$$\underline{\phi}_{End}^{B_{Strt}} = \underline{\phi}^{B_{Strt}} \quad @ \quad t = t_{End}$$

6.0 MEASUREMENT ERROR MODEL FOR THE IMPROVED SRT

The error model used for SRT determination of IMU calibration coefficient error is a linearized simplified version of (43) - (45) and (64).

The linearized form of (43) becomes for $\hat{a}^{B_{Strt}}_{Strt_{Dwn}}$:

$$\hat{a}^{B_{Strt}}_{Strt_{Dwn}} \approx \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \cdot \delta\underline{\hat{a}}^{B_{Strt}}_{SF_{Strt}} \tag{65}$$

For $\hat{a}^{B_{Strt}}_{End_{Dwn}}$, a more useful form of linearized (44) is based on the invariant property of vector dot products making them identical when evaluated in any coordinate frame. Thus $\hat{a}^{B_{Strt}}_{End_{Dwn}}$ in (44) approximates as:

$$\hat{a}^{B_{Strt}}_{End_{Dwn}} \approx \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \cdot \left(C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \, \delta\underline{\hat{a}}^{B_{End}}_{SF_{End}} \right) = \underline{u}^{B^{Nom}_{End}}_{Dwn} \cdot \delta\underline{\hat{a}}^{B_{End}}_{SF_{End}} \tag{66}$$

The $\Delta\underline{\hat{a}}^{B_{Strt}}_{H}$ measurement model for the improved SRT is a horizontal linearized approximation to (45) in which second order terms are deleted. Recognizing that because $\underline{u}^{B^{Nom}_{Strt}}_{Dwn}$ is vertical, $\underline{u}^{B^{Nom}_{Strt}}_{Dwn} \times \underline{\phi}^{B_{Strt}}_{End}$ is horizontal, hence, the linearized horizontal component of (45) becomes:

$$\Delta\underline{\hat{a}}^{B_{Strt}}_{H} \approx g \, \underline{u}^{B^{Nom}_{Strt}}_{Dwn} \times \underline{\phi}^{B_{Strt}}_{End} + \left(C^{B^{Nom}_{Strt}}_{B^{Nom}_{End}} \, \delta\underline{\hat{a}}^{B_{End}}_{SF_{End}} - \delta\underline{\hat{a}}^{B_{Strt}}_{SF_{Strt}} \right)_{H} \tag{67}$$

Section 6.1 (to follow) leading to (90) shows that by neglecting second order terms, earth rate effects, gyro bias, and gyro noise, $\underline{\phi}^{B_{Strt}}_{End}$ in (67) can be approximated as

$$\underline{\phi}^{B_{Strt}}_{End} = \sum_{i} C^{B^{Nom}_{Strt}}_{B^{Nom}_{i,Strt}} \left\{ \begin{array}{c} \left[\kappa_{LinScal} + \kappa_{Asym} \, \mathrm{Sign}\left(\dot{\beta}_i\right) \right] \underline{u}^{B^{Nom}_{i,Strt}}_{i} \, \theta_i \\ + \left[I \sin\theta_i + \left(1 - \cos\theta_i\right) \left(\underline{u}^{B^{Nom}_{i,Strt}}_{i} \times \right) \right] \left(\kappa_{Mis} \, \underline{u}^{B^{Nom}_{i,Strt}}_{i} \right) \end{array} \right\} \tag{68}$$

By neglecting second order terms and accelerometer noise, the $\delta\underline{\hat{a}}^{B_{End}}_{SF_{End}}$, $\delta\underline{\hat{a}}^{B_{Strt}}_{SF_{Strt}}$ terms in (65) - (67) are from (46) and (47):

$$\begin{aligned}
\delta \underline{\hat{a}}_{SF\,Strt}^{B\,Strt} &\approx -g\left(\lambda_{LinScal/Mis} + \lambda_{Asym}\, A_{SFSign}^{B\,Strt}\right) \underline{u}_{Dwn}^{B_{Strt}^{Nom}} + \underline{\lambda}_{Bias} \\
\delta \underline{\hat{a}}_{SF\,End}^{B\,End} &\approx -g\left(\lambda_{LinScal/Mis} + \lambda_{Asym}\, A_{SFSign}^{B\,End}\right) \underline{u}_{Dwn}^{B_{End}^{Nom}} + \underline{\lambda}_{Bias}
\end{aligned} \tag{69}$$

In summary, the error model equivalents to the (15) measurements are the approximate (65) - (69) expressions. Consistent with the approximation in these equations and linearization, the B^{Nom} frames can be approximated as B frames with which the equations simplify to the final form for SRT application:

$$\underline{\phi}_{End}^{B\,Strt} = \sum_i C_{B_{i,Strt}}^{B\,Strt} \left\{ \begin{array}{c} \left[\kappa_{LinScal} + \kappa_{Asym}\,\mathrm{Sign}\left(\dot{\beta}_i\right)\right] \underline{u}_i^{B_{i,Strt}}\, \theta_i \\ + \left[I \sin\theta_i + \left(1-\cos\theta_i\right)\left(\underline{u}_i^{B_{i,Strt}} \times\right)\right] \left(\kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}}\right) \end{array} \right\} \tag{70}$$

$$\begin{aligned}
\delta \underline{\hat{a}}_{SF\,Strt}^{B\,Strt} &\approx -g\left(\lambda_{LinScal/Mis} + \lambda_{Asym}\, A_{SFSign}^{B\,Strt}\right) \underline{u}_{Dwn}^{B\,Strt} + \underline{\lambda}_{Bias} \\
\delta \underline{\hat{a}}_{SF\,End}^{B\,End} &\approx -g\left(\lambda_{LinScal/Mis} + \lambda_{Asym}\, A_{SFSign}^{B\,End}\right) \underline{u}_{Dwn}^{B\,End} + \underline{\lambda}_{Bias}
\end{aligned} \tag{71}$$

$$\begin{aligned}
&\Delta \underline{\hat{a}}_{H}^{B\,Strt} \approx g\, \underline{u}_{Dwn}^{B\,Strt} \times \underline{\phi}_{End}^{B\,Strt} + \left(C_{B\,End}^{B\,Strt}\, \delta \underline{\hat{a}}_{SF\,End}^{B\,End} - \delta \underline{\hat{a}}_{SF\,Strt}^{B\,Strt} \right)_H \\
&\hat{a}_{Strt\,Dwn}^{B\,Strt} \approx \underline{u}_{Dwn}^{B\,Strt} \cdot \delta \underline{\hat{a}}_{SF\,Strt}^{B\,Strt} \qquad \hat{a}_{End\,Down}^{B\,Strt} \approx \underline{u}_{Dwn}^{B\,End} \cdot \delta \underline{\hat{a}}_{SF\,End}^{B\,End}
\end{aligned} \tag{72}$$

The $C_{B_{i,Strt}}^{B\,Strt}$ and $C_{B\,End}^{B\,Strt}$ matrices in (70) and (72) would be calculated for each rotation sequence using the recursive form

$$\begin{aligned}
&C_{B_{1,Strt}}^{B\,Strt} = I \\
&\text{Do } \; i = 1 \text{ To } n \\
&\qquad C_{B_{i+1,Strt}}^{B\,Strt} = C_{B_{i,Strt}}^{B\,Strt}\, C_{B_{i+1,Strt}}^{B_{i,Strt}} \\
&\text{End Do} \\
&C_{B\,End}^{B\,Strt} = C_{B_{n+1,Strt}}^{B\,Strt}
\end{aligned} \tag{73}$$

where

n = Total number of rotations in the rotation sequence.

The $C_{B_{i+1,Strt}}^{B_{i,Strt}}$ matrix in (73) can be determined from Part 1 [3, Table 1] for each rotation in the sequence using (84) (in Section 6.1 to follow), with the general B^{Nom} orientation specialized to the starting orientation for the next rotation, and the general the β_i rotation angle during rotation *i* in (84), replaced by the total traversal angle θ_i for that rotation:

$$C_{B_{i+1,Strt}}^{B_{i,Strt}} = I + \sin\theta_i \left(\underline{u}_i^{B_{i,Strt}} \times \right) + (1 - \cos\theta_i) \left(\underline{u}_i^{B_{i,Strt}} \times \right)^2 \tag{74}$$

The $\underline{u}_{Dwn}^{B_{End}}$ vector in (71) would be calculated as

$$\underline{u}_{Dwn}^{B_{End}} = \left(C_{B_{End}}^{B_{Strt}} \right)^T \underline{u}_{Dwn}^{B_{Strt}} \tag{75}$$

6.1 DERIVATION OF THE $\underline{\phi}_{End}^{B_{Strt}}$ ERROR MODEL FOR THE SRT

This section derives the (70) simplified formula for $\underline{\phi}_{End}^{B_{Strt}}$ used in the SRT. The derivation is based on second order $\underline{\phi}_{End}^{B_{Strt}}$ equation (64) in which the following rationale is applied to eliminate terms in $\dot{\underline{\phi}}^{B_{Strt}}$ as negligible for SRT rotation sequence execution on a two-axis rotation fixture with a nominal IMU mounting.

The integrated effect of $\dot{\underline{\alpha}}^{B^{Nom}}$ in $\dot{\underline{\phi}}^{B_{Strt}}$ of Eq. (64) over an SRT rotation sequence is bounded, and on the order of $\underline{\alpha}^{B^{Nom}}$. Because the *B* and B^{Nom} frames approximately overlap during the rotation sequence, $\underline{\alpha}^{B^{Nom}}$ will be small, hence, the second and third lines in $\dot{\underline{\phi}}^{B_{Strt}}$, and $\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{\omega}_{I:E}^{B_{Strt}^{Nom}}$ in the fourth line of $\dot{\underline{\phi}}^{B_{Strt}}$, will be negligible compared to the first line.

Because each SRT rotation sequence is of short duration (e.g., 10 seconds for rotations plus 10 seconds each for the initial and final acceleration measurements), the integrated effect of $\underline{\omega}_{I:E}^{B^{Nom}}$ in the first line of $\dot{\underline{\phi}}^{B_{Strt}}$ will be generally negligible compared to the integrated effect of $\underline{\omega}_{E:B^{Nom}}^{B^{Nom}}$. Because the integral of the first line in $\dot{\underline{\phi}}^{B_{Strt}}$ dominates short term change in $\underline{\phi}^{B_{Strt}}$, and because the integral of $\underline{\omega}_{E:B^{Nom}}^{B^{Nom}}$ due to rotation is on the order of π, $\underline{\phi}^{B_{Strt}}$ during a rotation sequence will be on the order of $\left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \Omega_{EBSign}^{B} \right) \pi$. Thus, since

$\underline{\alpha}^{B^{Nom}}$ and κ_{Mis} are of the same order of magnitude (e.g., 1 milli-radian), and since $\underline{\omega}_{I:E}^{B^{Nom}}$ is small compared to $\underline{\omega}_{E:B^{Nom}}^{B^{Nom}}$, the $\underline{\phi}^{B_{Strt}} \times \underline{\omega}_{I:E}^{B_{Strt}}$ term in the fourth line of $\dot{\underline{\phi}}^{B_{Strt}}$ will be negligible compared to $\left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \Omega_{EBSign}^{B}\right) \underline{\omega}_{E:B^{Nom}}^{B^{Nom}}$ in the first line.

Assuming that the gyro bias is calibrated to a reasonable accuracy prior to the rotation test, the $\underline{\kappa}_{Bias}$ term in $\dot{\underline{\phi}}^{B_{Strt}}$ can be ignored. Finally, because of the short time interval for a rotation sequence, the integrated effect of the noise terms in $\dot{\underline{\phi}}^{B_{Strt}}$ will be negligibly small.

Based on the previous rationale, all but the leading $C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \Omega_{EBSign}^{B}\right) \underline{\omega}_{E:B^{Nom}}^{B^{Nom}}$ term in $\dot{\underline{\phi}}^{B_{Strt}}$ can be neglected, yielding a simplified (64) result for $\underline{\phi}_{End}^{B_{Strt}}$:

$$
\begin{aligned}
\underline{\phi}^{B_{Strt}} &= \int_{t_{Strt}}^{t} \dot{\underline{\phi}}^{B_{Strt}}\, dt \\
\dot{\underline{\phi}}^{B_{Strt}} &\approx C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \Omega_{EBSign}^{B}\right) \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \\
\underline{\phi}_{End}^{B_{Strt}} &= \underline{\phi}^{B_{Strt}} \quad @ \quad t = t_{End}
\end{aligned}
\tag{76}
$$

Equations (76) represent the error in the (16) integration formula for $\hat{C}_{B}^{B_{Strt}}$. It remains to find an analytical solution for the (76) integral under typical SRT rotation sequence characteristics.

First, we note that $\dot{\underline{\phi}}^{B_{Strt}}$ in (76) is a linear differential equation whose integral satisfies the principle of linear superposition. Thus, $\underline{\phi}_{End}^{B_{Strt}}$ in (76) can be defined as the sum of individual angular errors generated during each rotation in a particular rotation sequence:

$$
\underline{\phi}_{End}^{B_{Strt}} = \sum_{i} \Delta \underline{\phi}_{i}^{B_{Strt}} \tag{77}
$$

where

$\Delta \underline{\phi}_{i}^{B_{Strt}}$ = Integral of $\dot{\underline{\phi}}^{B_{Strt}}$ for rotation i in the rotation sequence.

Equation (77) applies for any segment of the rotation sequence, hence from (76):

$$\Delta \underline{\dot{\phi}}_i^{B\,Strt} = C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \Omega_{EBSign}^{B} \right) \left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right)_i \quad (78)$$

where

$\left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right)_i$ = Angular rate vector $\underline{\omega}_{E:B^{Nom}}^{B^{Nom}}$ for the i^{th} rotation in the rotation sequence.

We now restrict each rotation in a rotation sequence to be around one of the IMU nominal B frame axes. Under this restriction, $\left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right)_i$ in (78) will be

$$\left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right)_i = \dot{\beta}_i\, \underline{u}_i^{B^{Nom}} \quad (79)$$

where

$\underline{u}_i^{B^{Nom}}$ = Unit vector along $\left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right)_i$ which is now specialized to lie along a particular B^{Nom} frame axis (i.e., along B^{Nom} frame axis x, y, or z) for the i^{th} rotation in the sequence.

$\dot{\beta}_i$ = Signed magnitude of $\left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right)_i$ defined as the projection of $\left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \right)_i$ on $\underline{u}_i^{B^{Nom}}$.

Substituting (79) in (78) gives

$$\Delta \underline{\dot{\phi}}_i^{B\,Strt} = C_{B^{Nom}}^{B_{Strt}^{Nom}}\, \kappa_i\, \dot{\beta}_i\, \underline{u}_i^{B^{Nom}} \quad (80)$$

where

$$\kappa_i \equiv \kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \mathrm{Sign}\left(\dot{\beta}_i \right) \quad (81)$$

The $C_{B^{Nom}}^{B_{Strt}^{Nom}}$ term in (80) during the i^{th} rotation can be written as

$$C_{B^{Nom}}^{B_{Strt}^{Nom}} = C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}}\, C_{B^{Nom}}^{B_{i,Strt}^{Nom}} \quad (82)$$

where

$B_{i,\,Strt}^{Nom}$ = B^{Nom} frame at the start of the i^{th} rotation in the sequence.

$C_{B^{Nom}}^{B_{i,Strt}^{Nom}}$ = Direction cosine matrix that transforms vectors from their B^{Nom} frame value during rotation i to their B^{Nom} frame value at the start of rotation i.

For the constant $\underline{u}_i^{B^{Nom}}$ rotation axis during rotation i, $C_{B^{Nom}}^{B_{i,Strt}^{Nom}}$ can be written from generalized Eq. (27) as

$$\underline{\beta}_i^{B^{Nom}} = \beta_i \, \underline{u}_i^{B^{Nom}}$$

$$C_{B^{Nom}}^{B_{i,Strt}^{Nom}} = I + \frac{\sin\beta_i}{\beta_i}\left(\underline{\beta}_i^{B^{Nom}} \times\right) + \frac{\left(1-\cos\beta_i\right)}{\beta_i^2}\left(\underline{\beta}_i^{B^{Nom}} \times\right)^2 \tag{83}$$
$$= I + \sin\beta_i\left(\underline{u}_i^{B^{Nom}} \times\right) + \left(1-\cos\beta_i\right)\left(\underline{u}_i^{B^{Nom}} \times\right)^2$$

where

β_i = The integral of $\dot{\beta}_i$ from the start of rotation i to a general time in the rotation.

$\underline{\beta}_i^{B^{Nom}}$ = Rotation vector equivalent of $C_{B^{Nom}}^{B_{i,Strt}^{Nom}}$.

Since $\underline{u}_i^{B^{Nom}}$ is constant, it equals $\underline{u}_i^{B_{i,Strt}^{Nom}}$, hence, (83) is equivalently

$$C_{B^{Nom}}^{B_{i,Strt}^{Nom}} = I + \sin\beta_i\left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times\right) + \left(1-\cos\beta_i\right)\left(\underline{u}_i^{B_{i,\,Strt}^{Nom}} \times\right)^2 \tag{84}$$

Substituting (82) with (84) in (80) obtains with rearrangement:

$$\Delta \dot{\underline{\phi}}_i^{B_{Strt}} = C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}} \left[I + \sin\beta_i \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right) + \left(1 - \cos\beta_i\right) \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right)^2 \right] \kappa_i \, \dot{\beta}_i \, \underline{u}_i^{B_{i,Strt}^{Nom}}$$

$$= C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}} \left\{ \begin{array}{l} \kappa_i \, \underline{u}_i^{B_{i,Strt}^{Nom}} + \left[\underline{u}_i^{B_{i,Strt}^{Nom}} \times \left(\kappa_i \;\; \underline{u}_i^{B_{i,Strt}^{Nom}} \right) \right] \sin\beta_i \\ + \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right)^2 \left(\kappa_i \;\; \underline{u}_i^{B_{i,Strt}^{Nom}} \right) \left(1 - \cos\beta_i\right) \end{array} \right\} \dot{\beta}_i \tag{85}$$

Recognizing that $\dot{\beta}_i \, dt = d\,\beta_i$, (85) can be integrated over the range from $\beta_i = 0$ through the total angular traversal of β_i. Equating the result to $\Delta \underline{\phi}_i^{B_{Strt}}$ in (77) then obtains

$$\Delta \underline{\phi}_i^{B_{Strt}} = C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}} \left\{ \begin{array}{c} \left[I + \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right)^2 \right] \kappa_i \, \underline{u}_i^{B_{i,Strt}^{Nom}} \, \theta_i - \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right)^2 \kappa_i \, \underline{u}_i^{B_{i,Strt}^{Nom}} \sin\theta_i \\ + \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right) \kappa_i \, \underline{u}_i^{B_{i,Strt}^{Nom}} \left(1 - \cos\theta_i\right) \end{array} \right\} \tag{86}$$

where

θ_i = Signed magnitude of the total angular traversal around rotation axis *i*.

Equation (86) can be simplified when the properties of $\kappa_i \, \underline{u}_i^{B_{i,Strt}^{Nom}}$ in (86) are taken into account. From (81), $\kappa_i \, \underline{u}_i^{B_{i,Strt}^{Nom}}$ is

$$\kappa_i \, \underline{u}_i^{B_{i,Strt}^{Nom}} = \left[\kappa_{LinScal} + \kappa_{Asym} \, \mathrm{Sign}\left(\dot{\beta}_i\right) \right] \underline{u}_i^{B_{i,Strt}^{Nom}} + \kappa_{Mis} \, \underline{u}_i^{B_{i,Strt}^{Nom}} \tag{87}$$

Because rotation *i* is about a single $B_{i,\,Strt}^{Nom}$ frame axis (i.e., about *x*, *y*, or *z*), Part 1 [3, Eqs. (8) or (14)] show that $\left[\kappa_{LinScal} + \kappa_{Asym} \, \mathrm{Sign}\left(\dot{\beta}_i\right) \right] \underline{u}_i^{B_{i,Strt}^{Nom}}$ in (87) is along $\underline{u}_i^{B_{i,Strt}^{Nom}}$, hence, its cross-product with $\underline{u}_i^{B_{i,Strt}^{Nom}}$ will be zero. Additionally, because rotation *i* is around a single $B_{i,\,Strt}^{Nom}$ frame axis, Part 1 [3, Eqs. (8) or (14)] show that $\kappa_{Mis} \, \underline{u}_i^{B_{i,Strt}^{Nom}}$ in (87) will have no component along $\underline{u}_i^{B_{i,Strt}^{Nom}}$, i.e., $\underline{u}_i^{B_{i,Strt}^{Nom}} \cdot \left(\kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}^{Nom}} \right) = 0$. Thus, $\left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right)^2 \kappa_i\, \underline{u}_i^{B_{i,Strt}^{Nom}}$ in (86) with (87) simplifies to

$$
\begin{aligned}
\left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right)^2 \kappa_i\, \underline{u}_i^{B_{i,Strt}^{Nom}} &= \underline{u}_i^{B_{i,Strt}^{Nom}} \times \left[\underline{u}_i^{B_{i,Strt}^{Nom}} \times \left(\kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}^{Nom}} \right) \right] \\
&= \underline{u}_i^{B_{i,Strt}^{Nom}} \left[\underline{u}_i^{B_{i,Strt}^{Nom}} \cdot \left(\kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}^{Nom}} \right) \right] - \kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}^{Nom}} = -\, \kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}^{Nom}}
\end{aligned}
\tag{88}
$$

With (87) and (88), (86) becomes:

$$
\begin{aligned}
\Delta\underline{\phi}_i^{B_{Strt}} &= C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}} \left\{ \begin{array}{c} \left[\kappa_{LinScal} + \kappa_{Asym}\, \mathrm{Sign}\left(\dot{\beta}_i \right) \right] \theta_i + \left(\kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}^{Nom}} \right) \sin\theta_i \\ + \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right) \left(\kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}^{Nom}} \right) \left(1 - \cos\theta_i \right) \end{array} \right\} \\
&= C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}} \left\{ \begin{array}{c} \left[\kappa_{LinScal} + \kappa_{Asym}\, \mathrm{Sign}\left(\dot{\beta}_i \right) \right] \underline{u}_i^{B_{i,Strt}^{Nom}}\, \theta_i \\ + \left[I \sin\theta_i + \left(1 - \cos\theta_i \right) \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right) \right] \left(\kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}^{Nom}} \right) \end{array} \right\}
\end{aligned}
\tag{89}
$$

Substituting (89) into (77) then obtains the desired $\underline{\phi}_{End}^{B_{Strt}}$ expression for (67):

$$
\underline{\phi}_{End}^{B_{Strt}} = \sum_i C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}} \left\{ \begin{array}{c} \left[\kappa_{LinScal} + \kappa_{Asym}\, \mathrm{Sign}\left(\dot{\beta}_i \right) \right] \underline{u}_i^{B_{i,Strt}^{Nom}}\, \theta_i \\ + \left[I \sin\theta_i + \left(1 - \cos\theta_i \right) \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right) \right] \left(\kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}^{Nom}} \right) \end{array} \right\}
\tag{90}
$$

7.0 SRT MEASUREMENT MODEL APPROXIMATION ERRORS

Equations (70) - (72) for SRT sensor error determination are linearized approximations to the (43) - (45) and (64) second error models based on neglecting second order terms (products of sensor errors), rotation fixture imperfections in executing rotations, IMU mounting anomalies on the test fixture (relative to vertical and north), gyro bias variations from initial calibration, and sensor noise effects. This section analytically defines the inaccuracy induced by these approximations in determining sensor calibration coefficient errors with the improved SRT. Of particular interest is the impact of initial IMU uncertainty relative to north, the effect of gyro bias uncertainty, and errors induced by rotation fixture imperfections. These are the primary factors impacting SRT rotation fixture cost, accuracy, and test setup time/cost. The analysis in this section will be restricted to errors incurred using the $\Delta \underline{\hat{a}}^{B_{Strt}}$ acceleration difference measurement. The horizontal component of $\Delta \underline{\hat{a}}^{B_{Strt}}$ is the primary measurement used by the SRT in determining the principle sensor calibration error parameters impacting IMU accuracy.

The analysis begins with a revised notation version of the (72) SRT linearized error model for $\Delta \underline{\hat{a}}^{B_{Strt}}$:

$$\Delta \underline{\hat{a}}_0^{B_{Strt}} \equiv g\, \underline{u}_{Dwn}^{B_{Strt}^{Nom}} \times \underline{\phi}_{0_{End}}^{B_{Strt}} + C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, \delta \underline{\hat{a}}_{SF\,0_{End}}^{B_{End}} - \delta \underline{\hat{a}}_{SF\,0_{Strt}}^{B_{Strt}} \tag{91}$$

where

0 = Subscript indicating the linearized approximation parameters used in the (70) - (72) error models.

The linearized accelerometer error terms in (91) are defined from (71) as:

$$\begin{aligned} \delta \underline{\hat{a}}_{SF\,0_{Strt}}^{B_{Strt}} &\equiv -\, g\left(\lambda_{LinScal} + \lambda_{Mis} + \lambda_{Asym}\, A_{SFSign}^{B_{Strt}}\right) \underline{u}_{Dwn}^{B_{Strt}^{Nom}} + \underline{\lambda}_{Bias} \\ \delta \underline{\hat{a}}_{SF\,0_{End}}^{B_{End}} &\equiv -\, g\left(\lambda_{LinScal} + \lambda_{Mis} + \lambda_{Asym}\, A_{SFSign}^{B_{End}}\right) \underline{u}_{Dwn}^{B_{End}^{Nom}} + \underline{\lambda}_{Bias} \end{aligned} \tag{92}$$

The $\underline{\phi}_{0_{End}}^{B_{Strt}}$ term in (91) is defined similarly, but from the equivalent linearized form of (64), the basis for (70):

$$\begin{aligned} \underline{\phi}_0^{B_{Strt}} &= \int_{t_{Strt}}^{t} \dot{\underline{\phi}}_0^{B_{Strt}}\, dt \\ \dot{\underline{\phi}}_0^{B_{Strt}} &\equiv C_{B^{Nom}}^{B_{Strt}^{Nom}} \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\, \Omega_{EBSign}^{B}\right) \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} \\ \underline{\phi}_{0_{End}}^{B_{Strt}} &= \underline{\phi}_0^{B_{Strt}} \quad @ \quad t = t_{End} \end{aligned} \tag{93}$$

The full value (to second order accuracy) equivalent of (91) is from (45):

$$\begin{aligned} \Delta \underline{\hat{a}}^{B_{Strt}} = {} & g\left[\underline{u}_{Dwn}^{B_{Strt}^{Nom}} + \underline{u}_{Dwn}^{B_{Strt}^{Nom}} \times \left(\frac{1}{2}\underline{\phi}_{End}^{B_{Strt}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right)\right] \times \underline{\phi}_{End}^{B_{Strt}} + C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, \delta \underline{\hat{a}}_{SF\,End}^{B_{End}} - \delta \underline{\hat{a}}_{SF\,Strt}^{B_{Strt}} \\ & + \left[\left(\underline{\phi}_{End}^{B_{Strt}} + \underline{\alpha}_{End}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times\right] C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, \delta \underline{\hat{a}}_{SF\,End}^{B_{End}} \end{aligned} \tag{94}$$

Particular full value parameters in (94) can be equated to their approximation equivalents in (92) – (93) plus approximation error:

$$\delta\underline{\hat{a}}_{SF\,Strt}^{B\,Strt} = \delta\underline{\hat{a}}_{SF\,0_{Strt}}^{B\,Strt} + e\left(\delta\underline{\hat{a}}_{SF\,0_{Strt}}^{B\,Strt}\right) \qquad \delta\underline{\hat{a}}_{SF\,End}^{B\,End} = \delta\underline{\hat{a}}_{SF\,0_{End}}^{B\,End} + e\left(\delta\underline{\hat{a}}_{SF\,0_{End}}^{B\,Strt}\right)$$
$$\underline{\phi}_{End}^{B\,Strt} = \underline{\phi}_{0_{End}}^{B\,Strt} + e\left(\underline{\phi}_{0_{End}}^{B\,Strt}\right) \tag{95}$$

and the converse

$$e\left(\delta\underline{\hat{a}}_{SF\,0_{Strt}}^{B\,Strt}\right) = \delta\underline{\hat{a}}_{SF\,Strt}^{B\,Strt} - \delta\underline{\hat{a}}_{SF\,0_{Strt}}^{B\,Strt} \qquad e\left(\delta\underline{\hat{a}}_{SF\,0_{End}}^{B\,End}\right) = \delta\underline{\hat{a}}_{SF\,End}^{B\,End} - \delta\underline{\hat{a}}_{SF\,0_{End}}^{B\,End}$$
$$e\left(\underline{\phi}_{0_{End}}^{B\,Strt}\right) = \underline{\phi}_{End}^{B\,Strt} - \underline{\phi}_{0_{End}}^{B\,Strt} \tag{96}$$

with

$$e\left(\dot{\underline{\phi}}_{0}^{B\,Strt}\right) = \dot{\underline{\phi}}^{B\,Strt} - \dot{\underline{\phi}}_{0}^{B\,Strt} \qquad e\left(\underline{\phi}_{0}^{B\,Strt}\right) = \int_{t_{Strt}}^{t} e\left(\dot{\underline{\phi}}_{0}^{B\,Strt}\right) dt$$
$$e\left(\underline{\phi}_{0_{End}}^{B\,Strt}\right) = e\left(\underline{\phi}_{0}^{B\,Strt}\right) \;\; @ \;\; t = t_{End} \tag{97}$$

where

$e\left(\delta\underline{\hat{a}}_{SF\,0_{Strt}}^{B\,Strt}\right),\; e\left(\delta\underline{\hat{a}}_{SF\,0_{End}}^{B\,End}\right),\; e\left(\underline{\phi}_{0_{End}}^{B\,Strt}\right),\; e\left(\dot{\underline{\phi}}_{0}^{B\,Strt}\right)$ = Approximation error in the 0 subscripted (92) - (93) parameters.

Substituting (95) into (94) and neglecting $e(\;)$ products with $e(\;)$ and α , δ terms obtains

$$
\begin{aligned}
\Delta \hat{\underline{a}}^{B_{Strt}} &= g\left[\underline{u}_{Dwn}^{B_{Strt}^{Nom}} + \underline{u}_{Dwn}^{B_{Strt}^{Nom}} \times \left(\frac{1}{2}\underline{\phi}_{End}^{B_{Strt}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right)\right] \times \underline{\phi}_{End}^{B_{Strt}} + C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, \delta\hat{\underline{a}}_{SF_{End}}^{B_{End}} - \delta\hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}} \\
&\quad + \left[\left(\underline{\phi}_{End}^{B_{Strt}} + \underline{\alpha}_{End}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times\right] C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, \delta\hat{\underline{a}}_{SF_{End}}^{B_{End}} \\
&= g\left\langle \underline{u}_{Dwn}^{B_{Strt}^{Nom}} + \underline{u}_{Dwn}^{B_{Strt}^{Nom}} \times \left\{\frac{1}{2}\left[\underline{\phi}_{0_{End}}^{B_{Strt}} + e\left(\underline{\phi}_{0_{End}}^{B_{Strt}}\right)\right] - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right\}\right\rangle \times \left[\underline{\phi}_{0_{End}}^{B_{Strt}} + e\left(\underline{\phi}_{0_{End}}^{B_{Strt}}\right)\right] \\
&\quad + C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\left[\delta\hat{\underline{a}}_{SF_{0_{End}}}^{B_{End}} + e\left(\delta\hat{\underline{a}}_{SF_{0_{End}}}^{B_{End}}\right)\right] - \left[\delta\hat{\underline{a}}_{SF_{0_{Strt}}}^{B_{Strt}} + e\left(\delta\hat{\underline{a}}_{SF_{0_{Strt}}}^{B_{Strt}}\right)\right] \\
&\quad + \left[\left(\underline{\phi}_{0_{End}}^{B_{Strt}} + e\left(\underline{\phi}_{0_{End}}^{B_{Strt}}\right) + \underline{\alpha}_{End}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times\right] C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\left[\delta\hat{\underline{a}}_{SF_{0_{End}}}^{B_{End}} + e\left(\delta\hat{\underline{a}}_{SF_{0_{End}}}^{B_{Strt}}\right)\right] \\
&\approx g\, \underline{u}_{Dwn}^{B_{Strt}^{Nom}} \times \underline{\phi}_{0_{End}}^{B_{Strt}} + C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, \delta\hat{\underline{a}}_{SF_{0_{End}}}^{B_{End}} - \delta\hat{\underline{a}}_{SF_{0_{Strt}}}^{B_{Strt}} \\
&\quad + g\, \underline{u}_{Dwn}^{B_{Strt}^{Nom}} \times \left[e\left(\underline{\phi}_{0_{End}}^{B_{Strt}}\right) + \left(\frac{1}{2}\underline{\phi}_{0_{End}}^{B_{Strt}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times \underline{\phi}_{0_{End}}^{B_{Strt}}\right] \\
&\quad + C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, e\left(\delta\hat{\underline{a}}_{SF_{0_{End}}}^{B_{End}}\right) - e\left(\delta\hat{\underline{a}}_{SF_{0_{Strt}}}^{B_{Strt}}\right) + \left(\underline{\phi}_{0_{End}}^{B_{Strt}} + \underline{\alpha}_{End}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times \left(C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, \delta\hat{\underline{a}}_{SF_{0_{End}}}^{B_{End}}\right)
\end{aligned}
\tag{98}
$$

Substituting (91) in (98) with rearrangement then yields $\Delta\hat{\underline{a}}^{B_{Strt}}$ as a function of the neglected $e(\ )$ parameters:

$$
\Delta\hat{\underline{a}}^{B_{Strt}} = \Delta\hat{\underline{a}}_0^{B_{Strt}} + e\left(\Delta\hat{\underline{a}}_0^{B_{Strt}}\right) \tag{99}
$$

with

$$
\begin{aligned}
e\left(\Delta\hat{\underline{a}}_0^{B_{Strt}}\right) &= g\, \underline{u}_{Dwn}^{B_{Strt}^{Nom}} \times \left[e\left(\underline{\phi}_{0_{End}}^{B_{Strt}}\right) + \left(\frac{1}{2}\underline{\phi}_{0_{End}}^{B_{Strt}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times \underline{\phi}_{0_{End}}^{B_{Strt}}\right] \\
&\quad + C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, e\left(\delta\hat{\underline{a}}_{SF_{0_{End}}}^{B_{End}}\right) - e\left(\delta\hat{\underline{a}}_{SF_{0_{Strt}}}^{B_{Strt}}\right) + \left(\underline{\phi}_{0_{End}}^{B_{Strt}} + \underline{\alpha}_{End}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times \left(C_{B_{End}^{Nom}}^{B_{Strt}^{Nom}}\, \delta\hat{\underline{a}}_{SF_{0_{End}}}^{B_{End}}\right)
\end{aligned}
\tag{100}
$$

where

$e\left(\Delta\hat{\underline{a}}_0^{B_{Strt}}\right)$ = Approximation error in $\Delta\hat{\underline{a}}_0^{B_{Strt}}$ equation (91).

Analytical expressions for $e\left(\underline{\phi}_{0_{End}}^{B_{Strt}}\right)$, $e\left(\delta\underline{\hat{a}}_{SF_{0_{End}}}^{B_{Strt}}\right)$, and $e\left(\delta\underline{\hat{a}}_{SF_{0_{Strt}}}^{B_{Strt}}\right)$ in (100) are obtained similarly. The full value equivalents of $\delta\underline{\hat{a}}_{SF_{0_{End}}}^{B_{End}}$, $\delta\underline{\hat{a}}_{SF_{0_{Strt}}}^{B_{Strt}}$, $\underline{\dot{\phi}}_0^{B_{Strt}}$ in (92) - (93) are $\delta\underline{\hat{a}}_{SF_{End}}^{B_{End}}$, $\delta\underline{\hat{a}}_{SF_{Strt}}^{B_{Strt}}$, $\underline{\dot{\phi}}^{B_{Strt}}$ in (46), (47), and (63). Substituting (46), (47), (63), and (92) - (93) into (96) - (97) obtains $e\left(\delta\underline{\hat{a}}_{SF_{0_{Strt}}}^{B_{Strt}}\right)$, $e\left(\delta\underline{\hat{a}}_{SF_{0_{End}}}^{B_{Strt}}\right)$, and after rearrangement, $e\left(\underline{\phi}_{0_{End}}^{B_{Strt}}\right)$ for (99). With $\underline{\phi}^{B_{Strt}} \times \underline{\omega}_{I:E}^{B_{Strt}}$ in (63) approximated as $\left[\underline{\phi}_0^{B_{Strt}} + e\left(\underline{\phi}_0^{B_{Strt}}\right)\right] \times \underline{\omega}_{I:E}^{B_{Strt}} \approx \underline{\phi}_0^{B_{Strt}} \times \underline{\omega}_{I:E}^{B_{Strt}}$, the results are

$$
\begin{aligned}
e\left(\delta\underline{\hat{a}}_{SF_{0_{Strt}}}^{B_{Strt}}\right) &= g\left(\lambda_{LinScal/Mis} + \lambda_{Asym} A_{SFSign}^{B_{Strt}}\right)\left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) + \underline{\lambda}_{Quant_{Strt}} + \underline{\lambda}_{Rndm_{Strt}} \\
e\left(\delta\underline{\hat{a}}_{SF_{0_{End}}}^{B_{End}}\right) &= g\left(\lambda_{LinScal/Mis} + \lambda_{Asym} A_{SFSign}^{B_{End}}\right)\left(\underline{\alpha}_{End}^{B_{End}^{Nom}} \times \underline{u}_{Dwn}^{B_{End}^{Nom}}\right) + \underline{\lambda}_{Quant_{End}} + \underline{\lambda}_{Rndm_{End}}
\end{aligned}
\tag{101}
$$

$$
\begin{aligned}
e\left(\underline{\phi}_0^{B_{Strt}}\right) &= \int_{t_{Strt}}^{t} e\left(\underline{\dot{\phi}}_0^{B_{Strt}}\right) dt \\
e\left(\underline{\dot{\phi}}_0^{B_{Strt}}\right) &= C_{B^{Nom}}^{B_{Strt}^{Nom}}\left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\,\Omega_{EBSign}^{B}\right)\underline{\omega}_{I:E}^{B^{Nom}} \\
&+ \left(\underline{\alpha}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times \left[C_{B^{Nom}}^{B_{Strt}^{Nom}}\left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\,\Omega_{EBSign}^{B}\right)\underline{\omega}_{E:B^{Nom}}^{B^{Nom}}\right] \\
&+ C_{B^{Nom}}^{B_{Strt}^{Nom}}\left[\left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\,\Omega_{EBSign}^{B}\right)\left(\underline{\dot{\alpha}}^{B^{Nom}} - \underline{\alpha}^{B^{Nom}} \times \underline{\omega}_{E:B^{Nom}}^{B^{Nom}}\right)\right] \\
&+ C_{B^{Nom}}^{B_{Strt}^{Nom}}\left(\underline{\kappa}_{Bias} + \delta\underline{\omega}_{Quant} + \delta\underline{\omega}_{Rand}\right) + \left(\underline{\phi}_0^{B_{Strt}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right) \times \underline{\omega}_{I:E}^{B_{Strt}} \\
e\left(\underline{\phi}_{0_{End}}^{B_{Strt}}\right) &= e\left(\underline{\phi}_0^{B_{Strt}}\right) \quad @ \quad t = t_{End}
\end{aligned}
\tag{102}
$$

To summarize, $\Delta\underline{\hat{a}}^{B_{Strt}}$ in (99) with (100) represents the actual measurement taken during an SRT rotation sequence showing the $e\left(\Delta\underline{\hat{a}}_0^{B_{Strt}}\right)$ variation from the approximate $\Delta\underline{\hat{a}}_0^{B_{Strt}}$ measurement model in (91) - (93) used for IMU sensor calibration coefficient error determination. The $e\left(\underline{\phi}_{0_{End}}^{B_{Strt}}\right)$, $e\left(\delta\underline{\hat{a}}_{SF_{0_{End}}}^{B_{Strt}}\right)$, and $e\left(\delta\underline{\hat{a}}_{SF_{0_{Strt}}}^{B_{Strt}}\right)$ approximation error terms in (100) are defined in (101) - (102) as functions of sensor error parameters to be evaluated by the SRT, sensor error terms that were neglected in the linearization process, initial IMU orientation uncertainties, and rotation errors introduced by the rotation fixture in executing SRT rotation sequences.

8.0 EFFECT OF GYRO BIAS UNCERTAINTY ON SRT ACCURACY

A primary inaccuracy in SRT sensor error determination is caused by neglecting gyro bias in the derivation of (70) for $\underline{\phi}_{End}^{B\,Strt}$ used in the (72) $\Delta\hat{\underline{a}}_{H}^{B\,Strt}$ equation. The effect is $C_{B^{Nom}}^{B_{Strt}^{Nom}}\,\underline{\kappa}_{Bias}$ in (102) for $e\left(\dot{\underline{\phi}}_{0}^{B\,Strt}\right)$ leading to a $\underline{\phi}_{End}^{B\,Strt}$ error in (72) for $\Delta\hat{\underline{a}}_{H}^{B\,Strt}$. The impact on $\Delta\hat{\underline{a}}_{H}^{B\,Strt}$ depends on the $C_{B^{Nom}}^{B_{Strt}^{Nom}}$ profile during a rotation sequence, but also, on the time duration and method for making the SRT acceleration measurements in (7) when calculating $\Delta\hat{\underline{a}}_{H}^{B\,Strt}$.

This section develops a general equation for analyzing the gyro bias SRT induced error effect assuming no g-sensitivity in the gyro bias error model. The discussion is grouped into four parts: 1) How gyro bias affects IMU inertial acceleration under stationary conditions, 2) How gyro bias translates through averaging filters used in making the SRT acceleration measurements, 3) How the SRT calculation of $\Delta\hat{\underline{a}}_{H}^{B\,Strt}$ is modified by the gyro bias effect on measurement filter outputs, and 4) The effect of gyro bias on $\Delta\hat{\underline{a}}_{H}^{B\,Strt}$ when using a simple linear average or average-of-averages type filter.

8.1 IMPACT OF GYRO BIAS ON STATIONARY ACCELERATION

The analysis begins with stationary acceleration equation (5) and its truth model equivalent from (3):

$$\hat{\underline{a}}^{B\,Strt} = \hat{C}_{B}^{B\,Strt}\,\hat{\underline{a}}_{SF}^{B} + g\,\hat{C}_{NED}^{B\,Strt}\,\underline{u}_{Dwn}^{NED} \qquad (103)$$

$$\underline{a}^{B\,Strt} = C_{B}^{B\,Strt}\,\underline{a}_{SF}^{B} + g\,C_{NED}^{B\,Strt}\,\underline{u}_{Dwn}^{NED} = 0 \qquad (104)$$

Gyro bias errors only effect $\hat{C}_{B}^{B\,Strt}$ in (103). For gyro bias effect analysis we neglect errors in $\hat{\underline{a}}_{SF}^{B}$ and $\hat{C}_{NED}^{B\,Strt}$, defining:

$$\hat{C}_{B}^{B\,Strt} = C_{B}^{B\,Strt} + \delta\hat{C}_{B\,GyroBias}^{B\,Strt} \qquad \hat{\underline{a}}^{B\,Strt} = \hat{\underline{a}}_{GyroBias}^{B\,Strt} \qquad \hat{C}_{NED}^{B\,Strt} = C_{NED}^{B\,Strt} \qquad \hat{\underline{a}}_{SF}^{B} = \underline{a}_{SF}^{B} \qquad (105)$$

where

$\hat{\underline{a}}^{B\,Strt}_{GyroBias}$ = Component of $\hat{\underline{a}}^{B\,Strt}$ caused by gyro bias.

$\delta\hat{C}^{B\,Strt}_{B\,GyroBias}$ = Error in $\hat{C}^{B\,Strt}_{B}$ caused by gyro bias.

Substituting (105) into (103) obtains with (104):

$$\begin{aligned} \hat{\underline{a}}^{B\,Strt}_{GyroBias} &= \left(C^{B\,Strt}_{B} + \delta\hat{C}^{B\,Strt}_{B\,GyroBias}\right)\underline{a}^{B}_{SF} + g\; C^{B\,Strt}_{NED}\,\underline{u}^{NED}_{Dwn} \\ &= C^{B\,Strt}_{B}\;\underline{a}^{B}_{SF} + g\; C^{B\,Strt}_{NED}\,\underline{u}^{NED}_{Dwn} + \delta\hat{C}^{B\,Strt}_{B\,GyroBias}\;\underline{a}^{B}_{SF} = \delta\hat{C}^{B\,Strt}_{B\,GyroBias}\;\underline{a}^{B}_{SF} \end{aligned} \tag{106}$$

The $\delta\hat{C}^{B\,Strt}_{B\,GyroBias}$ error in (106) is by similarity to (39):

$$\delta\hat{C}^{B\,Strt}_{B\,GyroBias} \approx \left(\underline{\phi}^{B\,Strt}_{GyroBias}\times\right) C^{B\,Strt}_{B} \tag{107}$$

where

$\underline{\phi}^{B\,Strt}_{GyroBias}$ = Component of $\underline{\phi}^{B\,Strt}$, the rotation vector equivalent error in $\delta\hat{C}^{B\,Strt}_{B\,GyroBias}$ caused by gyro bias.

Equation (104) states that $C^{B\,Strt}_{B}\;\underline{a}^{B}_{SF} + g\; C^{B\,Strt}_{NED}\,\underline{u}^{NED}_{Dwn} = \underline{a}^{B\,Strt}_{SF} + g\;\underline{u}^{B\,Strt}_{Dwn} = 0$, hence,

$$\underline{a}^{B\,Strt}_{SF} = -g\;\underline{u}^{B\,Strt}_{Dwn} \tag{108}$$

Substituting (107) and (108) into (106) then obtains the final form:

$$\hat{\underline{a}}^{B\,Strt}_{GyroBias} = \left(\underline{\phi}^{B\,Strt}_{GyroBias}\times\right) C^{B\,Strt}_{B}\;\underline{a}^{B}_{SF} = \underline{\phi}^{B\,Strt}_{GyroBias}\times\underline{a}^{B\,Strt}_{SF} = -\,g\;\underline{\phi}^{B\,Strt}_{GyroBias}\times\underline{u}^{B\,Strt}_{Dwn} \tag{109}$$

The $\underline{\phi}^{B\,Strt}_{GyroBias}$ term in (109) is the integral of that portion of $\dot{\underline{\phi}}^{B\,Strt}$ in (58) caused by gyro bias, or with (B-23) from Appendix B for $\delta\hat{\underline{\omega}}^{B}_{I:B}$:

$$\dot{\underline{\phi}}^{B\,Strt}_{GyroBias} = C^{B\,Strt}_{B}\;\delta\hat{\underline{\omega}}^{B}_{I:B\,GyroBias} - \delta\hat{\underline{\omega}}^{B\,Strt}_{I:E} + \underline{\phi}^{B\,Strt}\times\underline{\omega}^{B\,Strt}_{I:E} \approx C^{B\,Strt}_{B}\;\underline{\kappa}_{Bias} \tag{110}$$

where

$\dot{\underline{\phi}}^{B\,Strt}_{GyroBias}$ = Portion of $\dot{\underline{\phi}}^{B\,Strt}$ caused by gyro bias.

$C_B^{B_{Strt}}$ = Direction cosine matrix that transforms vectors from the instantaneous IMU B frame axes into B_{Strt} coordinates.

During the first measurement, attitude is stationary at $C_B^{B_{Strt}} = I$, thus during the measurement, $\underline{\phi}_{GyroBias}^{B_{Strt}}$ in (109) is with (110):

$$\underline{\phi}_{GyroBias}^{B_{Strt}} = \int_{t_{Meas1Strt}}^{t} C_B^{B_{Strt}} \, \underline{\kappa}_{Bias} \, dt = \int_{t_{Meas1Strt}}^{t} \underline{\kappa}_{Bias} \, dt = \left(t - t_{Meas1Strt}\right) \underline{\kappa}_{Bias} \quad (111)$$

where

$t_{Meas1Strt}$ = Time t at the start of the first stationary SRT measurement.

During the second measurement the attitude is stationary at $C_B^{B_{Strt}} = C_{B_{Meas2Strt}}^{B_{Strt}}$, thus $\underline{\phi}_{GyroBias}^{B_{Strt}}$ in (109) during the measurement is with (110):

$$\begin{aligned}
\underline{\phi}_{GyroBias}^{B_{Strt}} &= \underline{\phi}_{GyroBias_{Meas2Strt}}^{B_{Strt}} + \int_{t_{Meas2Strt}}^{t} C_B^{B_{Strt}} \, \underline{\kappa}_{Bias} \, dt \\
&= \underline{\phi}_{GyroBias_{Meas2Strt}}^{B_{Strt}} + C_{B_{Meas2Strt}}^{B_{Strt}} \int_{t_{Meas2Strt}}^{t} \underline{\kappa}_{Bias} \, dt \\
&= \underline{\phi}_{GyroBias_{Meas2Strt}}^{B_{Strt}} + C_{B_{Meas2Strt}}^{B_{Strt}} \left(t - t_{Meas2Strt}\right) \underline{\kappa}_{Bias}
\end{aligned} \quad (112)$$

in which

$$\underline{\phi}_{GyroBias_{Meas2Strt}}^{B_{Strt}} = \underline{\phi}_{GyroBias_{Meas1End}}^{B_{Strt}} + \Delta\underline{\phi}_{GyroBias_{Rot}}^{B_{Strt}} \quad (113)$$

and where

$t_{Meas2Strt}$ = Time t at the start of the second (last) stationary measurement.

$B_{Meas2Strt}$ = B frame at the start of measurement 2.

$\underline{\phi}_{GyroBias_{Meas1End}}^{B_{Strt}}$ = $\underline{\phi}_{GyroBias}^{B_{Strt}}$ at the end of measurement 1.

$\Delta\underline{\phi}_{GyroBias_{Rot}}^{B_{Strt}}$ = Portion of $\underline{\phi}_{GyroBias}^{B_{Strt}}$ generated during the sequence rotations.

From (111), the $\Delta\underline{\phi}_{GyroBias_{Meas1}}^{B_{Strt}}$ term in (113) is

$$\Delta \underline{\phi}^{B\,Strt}_{GyroBias_{Meas1}} = T_{Meas}\, \underline{\kappa}_{Bias} \tag{114}$$

where

T_{Meas} = Time interval for each measurement.

Based on (77) and (110), $\Delta \underline{\phi}^{B\,Strt}_{GyroBias_{Rot}}$ in (113) is given by

$$\Delta \underline{\phi}^{B\,Strt}_{GyroBias_{Rot}} = \sum_i \Delta \underline{\phi}^{B\,Strt}_{i\,GyroBias} \qquad \Delta \underline{\phi}^{B\,Strt}_{i\,GyroBias} = \int_{t_{iStrt}}^{t_{iEnd}} \dot{\underline{\phi}}^{B\,Strt}_{GyroBias}\, dt = \int_{t_{iStrt}}^{t_{iEnd}} C_B^{B\,Strt}\, \underline{\kappa}_{Bias}\, dt \tag{115}$$

where

$\Delta \underline{\phi}^{B\,Strt}_{i\,GyroBias}$ = Portion of $\underline{\phi}^{B\,Strt}_{GyroBias}$ generated during rotation *i* in the rotation sequence.

t_{iStrt}, t_{iEnd} = Time at the start and end of rotation *i*.

8.2 STATIONARY ACCELERATION MEASUREMENTS

Acceleration measurements are made with the SRT by applying an averaging filter to the $\hat{C}_B^{B\,Strt}\, \hat{\underline{a}}_{SF}^{B}$ portion of $\hat{\underline{a}}^{B\,Strt}$ in (103) during the stationary time periods before and after executing sequence rotations. The $\hat{\underline{a}}^{B\,Strt}$ measurement is then obtained by adding the (103) gravity term to the filter output:

$$\begin{gathered} \hat{\underline{a}}^{B\,Strt}_{Meas} = \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left(\hat{C}_B^{B\,Strt}\, \hat{\underline{a}}_{SF}^{B}\right) dt + g\, \hat{C}_{NED}^{B\,Strt}\, \underline{u}_{Dwn}^{NED} \\ \text{With Normalization Constraint: } \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) dt = 1 \end{gathered} \tag{116}$$

where

$t_{Meas\,Strt}$, $t_{Meas\,End}$ = Time at the start and end of the $\hat{\underline{a}}^{B\,Strt}$ measurement time.

$\hat{\underline{a}}^{B\,Strt}_{Meas}$ = Measurement of $\hat{\underline{a}}^{B\,Strt}$ obtained with the averaging filter.

$\varsigma\left(t, t_{Meas\,End}\right)$ = Averaging filter weighting function, the filter response at $t_{Meas\,End}$ to a unit impulse input applied at time *t* during the measurement period. Typical averaging filters are a simple linear average and an average of successive overlapping averages (“average-of-averages”).

Applying (105) and (107) to (116) for gyro effects analysis, the methodology leading from (103) to (109) then finds

Normalization Constraint: $\int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) dt = 1$

$$
\begin{aligned}
\underline{\hat{a}}_{GyroBias_{Meas}}^{B\,Strt} &= \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left(\hat{C}_{B}^{B\,Strt}\, \underline{a}_{SF}^{B}\right) dt + g\, \underline{u}_{Dwn}^{B\,Strt} \\
&= \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left[\left(C_{B}^{B\,Strt} + \delta\hat{C}_{B\,GyroBias}^{B\,Strt}\right)\left(-g\, \underline{u}_{Dwn}^{B}\right) + g\, \underline{u}_{Dwn}^{B\,Strt}\right] dt \qquad (117) \\
&= \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \delta\hat{C}_{B\,GyroBias}^{B\,Strt} \left(-g\, \underline{u}_{Dwn}^{B}\right) dt \\
&= \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left(\underline{\phi}_{GyroBias}^{B\,Strt} \times\right) C_{B}^{B\,Strt} \left(-g\, \underline{u}_{Dwn}^{B}\right) dt \\
&= g\, \underline{u}_{Dwn}^{B\,Strt} \times \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \underline{\phi}_{GyroBias}^{B\,Strt}\, dt
\end{aligned}
$$

where

$\underline{\hat{a}}_{GyroBias_{Meas}}^{B\,Strt}$ = Portion of the $\underline{\hat{a}}_{Meas}^{B\,Strt}$ measurement produced by gyro bias.

Each measurement is taken at constant attitude, hence, $\underline{\phi}_{GyroBias}^{B\,Strt}$ changes from the integral of (110) as

$$
\begin{aligned}
\underline{\phi}_{GyroBias}^{B\,Strt} &= \underline{\phi}_{GyroBias_{MeasStrt}}^{B\,Strt} + \int_{t_{MeasStrt}}^{t} C_{B}^{B\,Strt}\, \underline{\kappa}_{Bias}\, dt \\
&\approx \underline{\phi}_{GyroBias_{MeasStrt}}^{B\,Strt} + C_{B_{Meas}}^{B\,Strt} \int_{t_{MeasStrt}}^{t} \underline{\kappa}_{Bias}\, dt \qquad (118) \\
&= \underline{\phi}_{GyroBias_{MeasStrt}}^{B\,Strt} + C_{B_{Meas}}^{B\,Strt} \left(t - t_{MeasStrt}\right) \underline{\kappa}_{Bias}
\end{aligned}
$$

where

$\underline{\phi}_{GyroBias_{MeasStrt}}^{B\,Strt}$ = Value of $\underline{\phi}_{GyroBias}^{B\,Strt}$ at the start of the measurement.

$C_{B_{Meas}}^{B\,Strt} = C_{B}^{B\,Strt}$ during the measurement.

Substituting (118) into (117) then finds for the impact of gyro bias on $\underline{\hat{a}}_{Meas}^{B\,Strt}$:

$$\underline{\hat{a}}^{B\,Strt}_{GyroBias_{Meas}} = g\,\underline{u}^{B\,Strt}_{Dwn} \times \left\{ \begin{array}{c} \underline{\phi}^{B\,Strt}_{GyroBias_{MeasStrt}} \\ + C^{B\,Strt}_{B_{Meas}} \left[\int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left(t - t_{MeasStrt}\right) dt \right] \underline{\kappa}_{Bias} \end{array} \right\} \quad (119)$$

$$= g\,\underline{u}^{B\,Strt}_{Dwn} \times \left(\underline{\phi}^{B\,Strt}_{GyroBias_{MeasStrt}} + F_{Meas}\,T_{Meas}\,C^{B\,Strt}_{B_{Meas}}\,\underline{\kappa}_{Bias} \right)$$

in which

$$F_{Meas} \equiv \frac{1}{T_{Meas}} \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left(t - t_{MeasStrt}\right) dt \qquad (120)$$

and where

T_{Meas} = Time interval for the measurement (from $t_{Meas\,Strt}$ to $t_{Meas\,End}$).

8.2.1 Making The First Acceleration Measurement

For the first acceleration measurement $C^{B\,Strt}_{B_{Meas}} = I$ and $\underline{\phi}^{B\,Strt}_{GyroBias_{MeasStrt}}$ = 0, hence, (119) becomes

$$\underline{\hat{a}}^{B\,Strt}_{GyroBias_{Meas1}} = g\,F_{Meas}\,T_{Meas}\,\underline{u}^{B\,Strt}_{Dwn} \times \underline{\kappa}_{Bias} \qquad (121)$$

where

$\underline{\hat{a}}^{B\,Strt}_{GyroBias_{Meas1}} = \underline{\hat{a}}^{B\,Strt}_{GyroBias_{Meas}}$ taken prior to sequence rotation execution.

At the end of the measurement (and start of the rotation sequence), (118) shows that

$$\underline{\phi}^{B\,Strt}_{GyroBias_{Meas1End}} = \underline{\phi}^{B\,Strt}_{GyroBias_{RotStrt}} = T_{Meas}\,\underline{\kappa}_{Bias} \qquad (122)$$

8.2.2 Making The Second Acceleration Measurement

At completion of the rotation sequence (and start of the second measurement), from (113) and (122),

$$\underline{\phi}^{B\,Strt}_{GyroBias_{RotEnd}} = \underline{\phi}^{B\,Strt}_{GyroBias_{RotStrt}} + \Delta\underline{\phi}^{B\,Strt}_{GyroBias_{Rot}} = T_{Meas}\,\underline{\kappa}_{Bias} + \Delta\underline{\phi}^{B\,Strt}_{GyroBias_{Rot}} \qquad (123)$$

where

$$\underline{\phi}^{B\,Strt}_{GyroBias\,RotEnd} \;=\; \underline{\phi}^{B\,Strt}_{GyroBias} \text{ at rotation sequence completion.}$$

For the second acceleration measurement, $\underline{\phi}^{B\,Strt}_{GyroBias\,RotEnd}$ in (123) is $\underline{\phi}^{B\,Strt}_{GyroBias\,Meas2Strt}$, i.e., $\underline{\phi}^{B\,Strt}_{GyroBias}$ at the start of measurement 2, thus (119) becomes

$$\begin{aligned}
\hat{\underline{a}}^{B\,Strt}_{GyroBias\,Meas2} &= g\,\underline{u}^{B\,Strt}_{Dwn} \times \left(\underline{\phi}^{B\,Strt}_{GyroBias\,Meas2Strt} + F_{Meas}\,T_{Meas}\,C^{B\,Strt}_{B\,Meas2}\,\underline{\kappa}_{Bias}\right) \\
&= g\,\underline{u}^{B\,Strt}_{Dwn} \times \left(T_{Meas}\,\underline{\kappa}_{Bias} + \Delta\underline{\phi}^{B\,Strt}_{GyroBias\,Rot} + F_{Meas}\,T_{Meas}\,C^{B\,Strt}_{B\,Meas2}\,\underline{\kappa}_{Bias}\right) \qquad (124) \\
&= g\,\underline{u}^{B\,Strt}_{Dwn} \times \left[\Delta\underline{\phi}^{B\,Strt}_{GyroBias\,Rot} + T_{Meas}\left(I + F_{Meas}\,C^{B\,Strt}_{B\,Meas2}\right)\underline{\kappa}_{Bias}\right]
\end{aligned}$$

where

$$\hat{\underline{a}}^{B\,Strt}_{GyroBias\,Meas2} \;=\; \hat{\underline{a}}^{B\,Strt}_{GyroBias\,Meas} \text{ taken at sequence rotation completion.}$$

$$C^{B\,Strt}_{B\,Meas2} \;=\; C^{B\,Strt}_{B\,Meas} \text{ attitude during the } \hat{\underline{a}}^{B\,Strt}_{GyroBias\,Meas2} \text{ measurement.}$$

8.3 CALCULATING THE SRT ACCELERATION DIFFERENCE MEASUREMENT

The SRT measurement used for sensor error determination is $\Delta\hat{\underline{a}}^{B\,Strt}$, the difference between acceleration measurements taken at the start and end of the rotation sequence. The portion caused by gyro bias is the difference between (124) and (121):

$$\begin{aligned}
\Delta\hat{\underline{a}}^{B\,Strt}_{GyroBias} &= \hat{\underline{a}}^{B\,Strt}_{GyroBias\,Meas2} - \hat{\underline{a}}^{B\,Strt}_{GyroBias\,Meas1} \\
&= g\,\underline{u}^{B\,Strt}_{Dwn} \times \begin{bmatrix} \Delta\underline{\phi}^{B\,Strt}_{GyroBias\,Rot} + T_{Meas}\left(I + F_{Meas}\,C^{B\,Strt}_{B\,Meas2}\right)\underline{\kappa}_{Bias} \\ - F_{Meas}\,T_{Meas}\,\underline{\kappa}_{Bias} \end{bmatrix}
\end{aligned} \qquad (125)$$

or with rearrangement

$$\Delta\hat{\underline{a}}^{B\,Strt}_{GyroBias} = g\,\underline{u}^{B\,Strt}_{Dwn} \times \left\{\Delta\underline{\phi}^{B\,Strt}_{GyroBias\,Rot} + T_{Meas}\left[I + F_{Meas}\left(C^{B\,Strt}_{B\,Meas2} - I\right)\right]\underline{\kappa}_{Bias}\right\} \qquad (126)$$

An analytical expression for the $\Delta\underline{\phi}_{GyroBias_{Rot}}^{B_{Strt}}$ term in (126) can be derived following the procedure that led to (90). Because $\dot{\underline{\phi}}_{GyroBias}^{B_{Strt}}$ in (110) is a linear differential equation, its integral for $\Delta\underline{\phi}_{GyroBias_{Rot}}^{B_{Strt}}$ in (126) satisfies the principle of linear superposition. Thus, $\Delta\underline{\phi}_{GyroBias_{Rot}}^{B_{Strt}}$ can be defined as the sum of individual angular errors generated during each rotation in a particular rotation sequence:

$$\Delta\underline{\phi}_{GyroBias_{Rot}}^{B_{Strt}} = \sum_i \Delta\underline{\phi}_{GyroBias_{Rot-i}}^{B_{Strt}} \tag{127}$$

where

$\Delta\underline{\phi}_{GyroBias_{Rot-i}}^{B_{Strt}}$ = Portion of $\Delta\underline{\phi}_{GyroBias_{Rot}}^{B_{Strt}}$ generated during rotation *i*.

Equation (110) applies for any segment of the rotation sequence, hence, $\Delta\underline{\phi}_{GyroBias_{Rot-i}}^{B_{Strt}}$ in (127) is

$$\begin{aligned}\Delta\underline{\phi}_{GyroBias_{Rot-i}}^{B_{Strt}} &= \int_{t_{Strt_i}}^{t_{End_i}} \dot{\underline{\phi}}_{GyroBias_{Rot-i}}^{B_{Strt}}\, dt = \int_{t_{Strt_i}}^{t_{End_i}} C_B^{B_{Strt}}\, \underline{\kappa}_{Bias}\, dt \approx \int_{t_{Strt_i}}^{t_{End_i}} C_{B^{Nom}}^{B_{Strt}^{Nom}}\, \underline{\kappa}_{Bias}\, dt \\ &= \int_{t_{Strt_i}}^{t_{End_i}} C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}}\, C_{B^{Nom}}^{B_{i,Strt}^{Nom}}\, \underline{\kappa}_{Bias}\, dt = C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}} \int_{t_{Strt_i}}^{t_{End_i}} C_{B^{Nom}}^{B_{i,Strt}^{Nom}}\, \underline{\kappa}_{Bias}\, dt\end{aligned} \tag{128}$$

where

t_{Strt_i}, t_{End_i} = Time *t* at the start and end of the i^{th} rotation in the sequence.

$B_{i,\,Strt}^{Nom}$ = B^{Nom} frame at the start of the i^{th} rotation in the sequence.

$C_{B^{Nom}}^{B_{i,Strt}^{Nom}}$ = Direction cosine matrix that transforms vectors from their B^{Nom} frame value during rotation *i* to their B^{Nom} frame value at the start of rotation *i*.

Restricting each rotation in a sequence to be around one of the IMU nominal *B* frame axes, $C_{B^{Nom}}^{B_{i,Strt}^{Nom}}$ can be written from generalized equation (27) as

$$C_{B^{Nom}}^{B_{i,Strt}^{Nom}} = I + \sin\beta_i \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right) + \left(1 - \cos\beta_i\right) \left(\underline{u}_i^{B_{i,\,Strt}^{Nom}} \times \right)^2 \qquad (129)$$

where

$\underline{u}_i^{B_{i,Strt}^{Nom}}$ = Unit vector along the *i* axis of rotation (in *B* frame axes at the start of rotation *i*).

β_i = Angular traversal of rotation *i* from rotation *i* start to a general time during rotation *i*.

Substituting $C_{B^{Nom}}^{B_{i,Strt}^{Nom}}$ from (129) in (127) while recognizing that $dt = d\beta_i \,/\, \dot{\beta}_i$ for the (127) integral, then obtains for $\Delta\underline{\phi}_{GyroBias_{Rot-i}}^{B\,Strt}$:

$$\Delta\underline{\phi}_{GyroBias_{Rot-i}}^{B\,Strt} = C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}} \left[I + \frac{(1 - \cos\theta_i)}{\theta_i} \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right) + \left(1 - \frac{\sin\theta_i}{\theta_i} \right) \left(\underline{u}_i^{B_{i,\,Strt}^{Nom}} \times \right)^2 \right] \frac{\theta_i}{\dot{\beta}_i} \,\underline{\kappa}_{Bias} \qquad (130)$$

where

θ_i = Signed magnitude of total angular traversal around rotation axis *i*.

$\dot{\beta}_i$ = Rate of change of β_i during rotation *i* (assumed constant).

Substituting (130) into (127) then finds for $\Delta\underline{\phi}_{GyroBias_{Rot}}^{B\,Strt}$ in (126)

$$\Delta\underline{\phi}_{GyroBias_{Rot}}^{B\,Strt} = \sum_i C_{B_{i,Strt}^{Nom}}^{B_{Strt}^{Nom}} \left[I + \frac{(1 - \cos\theta_i)}{\theta_i} \left(\underline{u}_i^{B_{i,Strt}^{Nom}} \times \right) + \left(1 - \frac{\sin\theta_i}{\theta_i} \right) \left(\underline{u}_i^{B_{i,\,Strt}^{Nom}} \times \right)^2 \right] \frac{\theta_i}{\dot{\beta}_i} \,\underline{\kappa}_{Bias} \qquad (131)$$

8.4 EVALUATING F_{Meas} FOR TWO AVERAGING FILTERS

This section derives F_{Meas} formulas for two commonly used averaging filters; a simple linear average and an average-of-averages.

For a linear averaging filter, the $\varsigma\left(t, t_{Meas\,End}\right)$ weighting function in (116) equals $1/T_{Meas}$ which satisfies the (116) normalization constraint: $\int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) dt = 1$. Thus from (120), F_{Meas} for a simple averaging filter becomes

$$
\begin{aligned}
F_{Meas} &\equiv \frac{1}{T_{Meas}} \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left(t - t_{MeasStrt}\right) dt \\
&= \frac{1}{T_{Meas}^2} \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \left(t - t_{MeasStrt}\right) dt = \frac{1}{2}
\end{aligned}
\tag{132}
$$

For an average-of-averages filter, the $\varsigma\left(t, t_{Meas\,End}\right)$ weighting function in (116) is given by [2, Sect. 18.4.7.3]:

$$
\begin{aligned}
&\text{If } \left[0 < \left(t - t_{MeasStrt}\right) \le \frac{T_{Meas}}{2}\right], \text{ Then: } \varsigma\left(t, t_{Meas\,End}\right) = \frac{4}{T_{Meas}^2}\left(t - t_{Meas\,End} + T_{Meas}\right) \\
&\text{If } \left[\frac{T_{Meas}}{2} < \left(t - t_{MeasStrt}\right) \le T_{Meas}\right], \text{ Then: } \varsigma\left(t, t_{Meas\,End}\right) = -\frac{4}{T_{Meas}^2}\left(t - t_{Meas\,End}\right)
\end{aligned}
\tag{133}
$$

which also satisfies the (116) normalization constraint: $\int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) dt = 1$. To evaluate F_{Meas} for an average-of-averages filter, we first convert (116) to an equivalent form that is more compatible with (133) notation:

$$
\begin{aligned}
F_{Meas} &\equiv \frac{1}{T_{Meas}} \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left(t - t_{MeasStrt}\right) dt \\
&= \frac{1}{T_{Meas}} \int_{t_{Meas\,Strt}}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left(t - t_{Meas\,End} + T_{Meas}\right) dt \\
&= \frac{1}{T_{Meas}} \int_{t_{Meas\,End} - T_{Meas}}^{t_{Meas\,End} - T_{Meas}/2} \varsigma\left(t, t_{Meas\,End}\right) \left(t - t_{Meas\,End} + T_{Meas}\right) dt \\
&\quad + \frac{1}{T_{Meas}} \int_{t_{Meas\,End} - T_{Meas}/2}^{t_{Meas\,End}} \varsigma\left(t, t_{Meas\,End}\right) \left(t - t_{Meas\,End} + T_{Meas}\right) dt
\end{aligned}
\tag{134}
$$

Substituting (133) for $\varsigma\left(t, t_{Meas\,End}\right)$ in (134) then obtains

$$
\begin{aligned}
F_{Meas} = {} & \frac{4}{T_{Meas}^3}\int_{t_{Meas\,End}-T_{Meas}}^{t_{Meas\,End}-T_{Meas}/2}\left(t - t_{Meas\,End} + T_{Meas}\right)^2 dt \\
& - \frac{4}{T_{Meas}^3}\int_{t_{Meas\,End}-T_{Meas}/2}^{t_{Meas\,End}}\left(t - t_{Meas\,End}\right)\left(t - t_{Meas\,End} + T_{Meas}\right) dt \\
= {} & \frac{4}{T_{Meas}^3}\int_{t_{Meas\,End}-T_{Meas}}^{t_{Meas\,End}-T_{Meas}/2}\left(t - t_{Meas\,End} + T_{Meas}\right)^2 dt \\
& - \frac{4}{T_{Meas}^3}\int_{t_{Meas\,End}-T_{Meas}/2}^{t_{Meas\,End}}\left(t - t_{Meas\,End}\right)^2 dt \\
& - \frac{4}{T_{Meas}^2}\int_{t_{Meas\,End}-T_{Meas}/2}^{t_{Meas\,End}}\left(t - t_{Meas\,End}\right) dt
\end{aligned}
\tag{135}
$$

For t at the (135) integration limits, the (135) bracketed terms are:

$$
\begin{gathered}
For\ t = t_{Meas\,End} - T_{Meas}/2: \\
t - t_{Meas\,End} + T_{Meas} = t_{Meas\,End} - T_{Meas}/2 - t_{Meas\,End} + T_{Meas} = T_{Meas}/2 \\
For\ t = t_{Meas\,End} - T_{Meas}: \\
t - t_{Meas\,End} + T_{Meas} = t_{Meas\,End} - T_{Meas} - t_{Meas\,End} + T_{Meas} = 0 \\
\\
For\ t = t_{Meas\,End}:\ t - t_{Meas\,End} = 0 \\
For\ t = t_{Meas\,End} - T_{Meas}/2: \\
t - t_{Meas\,End} = t_{Meas\,End} - T_{Meas}/2 - t_{Meas\,End} = -T_{Meas}/2
\end{gathered}
\tag{136}
$$

With the (136) terms at the integration limits, the integral terms in (135) become

$$
\begin{aligned}
& \frac{4}{T_{Meas}^3}\int_{t_{Meas\,End}-T_{Meas}}^{t_{Meas\,End}-T_{Meas}/2}\left(t - t_{Meas\,End} + T_{Meas}\right)^2 dt = \frac{4}{T_{Meas}^3}\frac{1}{3}\left(\frac{T_{Meas}}{2}\right)^3 = \frac{1}{6} \\
& \frac{4}{T_{Meas}^3}\int_{t_{Meas\,End}-T_{Meas}/2}^{t_{Meas\,End}}\left(t - t_{Meas\,End}\right)^2 dt = -\frac{4}{T_{Meas}^3}\frac{1}{3}\left(\frac{-T_{Meas}}{2}\right)^3 = \frac{1}{6} \\
& \frac{4}{T_{Meas}^2}\int_{t_{Meas\,End}-T_{Meas}/2}^{t_{Meas\,End}}\left(t - t_{Meas\,End}\right) dt = -\frac{4}{T_{Meas}^2}\frac{1}{2}\left(\frac{-T_{Meas}}{2}\right)^2 = -\frac{1}{2}
\end{aligned}
\tag{137}
$$

Substituting (137) into (135) then finds for the average-of-averages filter:

$$
F_{Meas} = \frac{1}{6} - \frac{1}{6} + \frac{1}{2} = \frac{1}{2} \tag{138}
$$

Curiously, F_{Meas} in (138) for the average-of-averages filter is the same as F_{Meas} in (132) for the simple linear averaging filter.

APPENDIX A

STRAPDOWN ACCELEROMETER TRIAD ANALYTICAL MODELS

Analytical models are developed in this appendix describing accelerometer triad uncompensated outputs in terms of specific force acceleration input, accelerometer triad compensation equations, and associated error models.

A.1 ACCELEROMETER TRIAD OUTPUT MODELS

The output vector from an uncompensated strapdown accelerometer triad can be characterized as a function of its specific force acceleration input vector by

$$\underline{a}_{SF_{Raw}} = \left(I + G_{Scal}\right)\left(G_{Algn}\ \underline{a}_{SF} + \delta\underline{a}_{Bias} + \delta\underline{a}_{Size} + \delta\underline{a}_{Aniso} + \delta\underline{a}_{Quant} + \delta\underline{a}_{Rndm}\right) \quad \text{(A-1)}$$

where

$\underline{a}_{SF_{Raw}}$ = Accelerometer triad uncompensated specific force acceleration output vector.

$\underline{a}_{SF}$ = Accelerometer triad specific force acceleration input vector.

I = Identity matrix.

G_{Scal} = Accelerometer triad scale factor diagonal error matrix. Nominally, the G_{Scal} matrix is zero. The G_{Scal} matrix may include non-linear scale factor effects and temperature dependency.

G_{Algn} = Accelerometer triad alignment matrix. Nominally, the G_{Algn} matrix is identity. The G_{Algn} matrix may include temperature dependency.

$\delta\underline{a}_{Bias}$ = Accelerometer triad bias vector. Each element equals the systematic output from a particular accelerometer under zero specific force acceleration input conditions. For some accelerometers, $\delta\underline{a}_{Bias}$ may have temperature and inertial angular rate sensitivities.

$\delta\underline{a}_{Size}$ = Accelerometer triad size effect error vector caused by accelerometers in the triad not being collocated, hence, not measuring components of identically the same acceleration vector (See [2, Sect. 8.1.4.1]).

$\delta \underline{a}_{Aniso}$ = Accelerometer triad anisoinertia error vector caused (in pendulous accelerometers) by mismatch in the moments of inertia around the input and pendulum axes (See [2, Sect. 8.1.4.2]).

$\delta \underline{a}_{Quant}$ = Accelerometer triad pulse quantization error vector caused by accelerometer outputs per axis only provided when the cumulative input equals the accelerometer pulse weight.

$\delta \underline{a}_{Rndm}$ = Accelerometer triad random output error vector.

An alternative form of the uncompensated accelerometer triad output model derives from (A-1) as follows:

$$
\begin{aligned}
\underline{a}_{SF\,Raw} &= \left(I + G_{Scal}\right)\left(G_{Algn}\ \underline{a}_{SF} + \delta \underline{a}_{Bias} + \delta \underline{a}_{Size} + \delta \underline{a}_{Aniso} + \delta \underline{a}_{Quant} + \delta \underline{a}_{Rndm}\right) \\
&= \left(I + G_{Scal}\right) G_{Algn}\ \underline{a}_{SF} + \left(I + G_{Scal}\right)\left(\delta \underline{a}_{Bias} + \delta \underline{a}_{Size} + \delta \underline{a}_{Aniso} + \delta \underline{a}_{Quant} + \delta \underline{a}_{Rndm}\right) \\
&= \underline{a}_{SF} + \left[\left(I + G_{Scal}\right) G_{Algn} - I\right] \underline{a}_{SF} \\
&\quad + \left(I + G_{Scal}\right)\left(\delta \underline{a}_{Bias} + \delta \underline{a}_{Size} + \delta \underline{a}_{Aniso} + \delta \underline{a}_{Quant} + \delta \underline{a}_{Rndm}\right) \\
&= \underline{a}_{SF} + \lambda_{Algn/Scal}\ \underline{a}_{SF} + \underline{\lambda}_{Bias} + \underline{\lambda}_{Size} + \underline{\lambda}_{Aniso} + \underline{\lambda}_{Quant} + \underline{\lambda}_{Rndm}
\end{aligned}
\tag{A-2}
$$

in which

$$
\begin{array}{lll}
\lambda_{Algn/Scal} \equiv \left(I + G_{Scal}\right) G_{Algn} - I & \underline{\lambda}_{Bias} \equiv \left(I + G_{Scal}\right) \delta \underline{a}_{Bias} & \underline{\lambda}_{Size} \equiv \left(I + G_{Scal}\right) \delta \underline{a}_{Size} \\
\underline{\lambda}_{Aniso} \equiv \left(I + G_{Scal}\right) \delta \underline{a}_{Aniso} & \underline{\lambda}_{Quant} \equiv \left(I + G_{Scal}\right) \delta \underline{a}_{Quant} & \underline{\lambda}_{Rndm} \equiv \left(I + G_{Scal}\right) \delta \underline{a}_{Rndm}
\end{array}
\tag{A-3}
$$

A.2 ACCELEROMETER TRIAD COMPENSATION ALGORITHMS

Compensation formulas to correct the (A-1) basic model output are structured based on the inverse of (A-1) with random noise terms deleted:

$$
\begin{aligned}
\underline{a}'_{SF} &= \left(I + L_{Scal}\right)^{-1} \underline{a}_{SF\,Raw} \\
\tilde{\underline{a}}_{SF} &= L_{Algn}^{-1} \left(\underline{a}'_{SF} - \underline{a}_{Bias} - \underline{a}_{Size} - \underline{a}_{Aniso} - \underline{a}_{Quant}\right)
\end{aligned}
\tag{A-4}
$$

where

$\tilde{\underline{a}}_{SF}$ = Compensated accelerometer triad output vector.

L_{Scal} = Accelerometer triad scale factor correction matrix, a diagonal matrix in which each element adjusts the output scaling to correspond to the actual scaling for the particular sensor output. Typically modeled by analytical equations containing premeasured coefficients. Nominally, the L_{Scal} matrix is zero. The L_{Scal} matrix may include non-linear scale factor effects and temperature dependency.

L_{Algn} = Accelerometer triad alignment correction matrix, typically modeled by a set of equations with premeasured coefficients. Nominally, the L_{Algn} matrix is identity. The L_{Algn} matrix may include temperature dependency.

$\underline{a}_{Bias}$ = Accelerometer triad bias correction vector with components typically modeled by analytical equations containing premeasured coefficients. Each element corrects the output of a particular accelerometer to zero under zero input specific force acceleration conditions. In some accelerometers, $\underline{a}_{Bias}$ may have temperature and angular rate sensitivities.

$\underline{a}_{Size}$ = Accelerometer triad size effect correction vector that compensates the error created by accelerometers in the triad not being collocated, hence, not measuring components of identically the same acceleration vector (See [2, Sect. 8.1.4.1]).

$\underline{a}_{Aniso}$ = Accelerometer triad anisoinertia correction vector that compensates for an error effect (in pendulous accelerometers) from mismatch in the moments of inertia around the input and pendulum axes (See [2, Sect. 8.1.4.2]).

$\underline{a}_{Quant}$ = Accelerometer triad pulse quantization correction vector for accelerometer outputs only being provided when the cumulative input equals the pulse weight per axis. Includes pulse output logic dead-band effect under turn-around conditions (See [2, Sect. 8.1.3.2]).

Similar to (A-4), a compensation formula can also be structured to correct the (A-2) alternative accelerometer model output as the inverse of (A-2) with the random errors excluded:

$$\hat{\underline{a}}_{SF} = \left(I + \lambda_{Algn/Scal}\right)^{-1} \left(\underline{a}_{SF_{Raw}} - \underline{\lambda}_{Bias} - \underline{\lambda}_{Size} - \underline{\lambda}_{Aniso} - \underline{\lambda}_{Quant}\right) \quad \text{(A-5)}$$

where

$\hat{\underline{a}}_{SF}$ = Compensated alternative accelerometer triad model output.

A.3 APPLICATION TO A TWO-STAGE COMPENSATION STRUCTURE

In many applications, a two-stage compensation structure is incorporated whereby (A-4) is used as the first stage compensation to correct accelerometer outputs, followed by (A-5) to correct residual errors in the (A-4) compensation. With such an approach, the $\underline{a}_{SF_{Raw}}$ input in (A-5) would represent the (A-4) output. Using this method, it is also assumed that the (A-4) compensation exactly corrects the size effect and anisoinertia errors (as design model characteristics), but leaves residual errors in the (A-4) output due to inaccuracy in determining the (A-4) compensation correction terms (L_{Algn}, L_{Scal}, $\underline{a}_{Bias}$, $\underline{a}_{Quant}$) and accelerometer changes since determination.

Note: The L_{Algn}, L_{Scal}, $\underline{a}_{Bias}$ term coefficients are typically measured for each individual sensor prior to installation in the IMU sensor assembly, and $\underline{a}_{Quant}$ may (or may not) be only approximately modeled as a design characteristic. Note also that the L_{Algn} matrix can only be accurately determined after installation in an IMU due to mounting uncertainties.

For a two-stage compensation structure, the alternative (A-2) model and its (A-5) compensation equation then become including coordinate frame designations:

$$\underline{\tilde{a}}_{SF}^{B} = \underline{a}_{SF}^{B} + \lambda_{Algn/Scal}\,\underline{a}_{SF}^{B} + \underline{\lambda}_{Bias} + \underline{\lambda}_{Quant} + \underline{\lambda}_{Rndm} \qquad \text{(A-6)}$$

$$\underline{\hat{a}}_{SF}^{B} = \left(I + \lambda_{Algn/Scal}\right)^{-1}\left(\underline{\tilde{a}}_{SF}^{B} - \underline{\lambda}_{Bias}\right) \qquad \text{(A-7)}$$

where

$\underline{\tilde{a}}_{SF}^{B}$ = Specific force acceleration output vector from the first stage (A-4) compensation equations in *B* frame coordinates, and containing compensation error residuals ($\tilde{\ }$ designation).

$\underline{a}_{SF}^{B}$ = True (error free) specific force acceleration vector in *B* frame coordinates.

$\underline{\hat{a}}_{SF}^{B}$ = Second stage compensated accelerometer triad output vector (in *B* frame coordinates) generated by second stage compensation (A-7) using first stage compensation (A-4) output as the (A-7) input.

$\lambda_{Algn/Scal}$ = Accelerometer triad alignment/scale-factor compensation error residual matrix.

$\underline{\lambda}_{Bias}$ = Accelerometer triad bias compensation error residual vector.

$\underline{\lambda}_{Quant}$ = Accelerometer triad output quantization compensation error residual vector.

$\underline{\lambda}_{Rndm}$ = Accelerometer triad output random noise vector.

A.4 ERROR MODELS FOR THE TWO-STAGE COMPENSATED ACCELEROMETER TRIAD

An error model for the two-stage compensated accelerometer triad output $\hat{\underline{a}}_{SF}^{B}$ is defined as the (A-7) output minus the true specific force acceleration $\underline{a}_{SF}^{B}$. For clarity, we first rewrite (A-7) as

$$\hat{\underline{a}}_{SF}^{B} = \left(I + \hat{\lambda}_{Algn/Scal}\right)^{-1} \left(\tilde{\underline{a}}_{SF}^{B} - \hat{\underline{\lambda}}_{Bias}\right) \tag{A-8}$$

using the $\hat{}$ notation to indicate that the error coefficient terms may still contain error. Then the error in (A-8) defines as

$$\delta\hat{\underline{a}}_{SF}^{B} \equiv \hat{\underline{a}}_{SF}^{B} - \underline{a}_{SF}^{B} = \left(I + \hat{\lambda}_{Algn/Scal}\right)^{-1} \left(\tilde{\underline{a}}_{SF}^{B} - \hat{\underline{\lambda}}_{Bias}\right) - \underline{a}_{SF}^{B} \tag{A-9}$$

where

$\delta\hat{\underline{a}}_{SF}^{B}$ = Error in $\hat{\underline{a}}_{SF}^{B}$. Error in the $\hat{\underline{a}}_{SF}^{B}$ output from (B-8), i.e., the error in the compensation estimate for $\underline{a}_{SF}^{B}$.

Substituting (A-6) for $\tilde{\underline{a}}_{SF}^{B}$ finds for the (A-9) output error:

$$\begin{aligned}
\delta\hat{\underline{a}}_{SF}^{B} &= \left(I + \hat{\lambda}_{Algn/Scal}\right)^{-1} \left(\tilde{\underline{a}}_{SF}^{B} - \hat{\underline{\lambda}}_{Bias}\right) - \underline{a}_{SF}^{B} \\
&= \left(I + \hat{\lambda}_{Algn/Scal}\right)^{-1} \left(\underline{a}_{SF}^{B} + \lambda_{Algn/Scal}\,\underline{a}_{SF}^{B} + \underline{\lambda}_{Bias} + \underline{\lambda}_{Quant} + \underline{\lambda}_{Rndm} - \hat{\underline{\lambda}}_{Bias}\right) - \underline{a}_{SF}^{B} \\
&= \left[\left(I + \hat{\lambda}_{Algn/Scal}\right)^{-1} \left(I + \lambda_{Algn/Scal}\right) - I\right] \underline{a}_{SF}^{B} \\
&\quad + \left(I + \hat{\lambda}_{Algn/Scal}\right)^{-1} \left(\underline{\lambda}_{Bias} - \hat{\underline{\lambda}}_{Bias} + \underline{\lambda}_{Quant} + \underline{\lambda}_{Rndm}\right)
\end{aligned} \tag{A-10}$$

An alternate form of (A-10) can be derived by first writing the $\underline{a}_{SF}^{B}$ coefficient in (A-10) in the equivalent form:

$$\left(I+\hat{\lambda}_{Algn/Scal}\right)^{-1}\left(I+\lambda_{Algn/Scal}\right)-I=\left[\left(I+\lambda_{Algn/Scal}\right)^{-1}\left(I+\hat{\lambda}_{Algn/Scal}\right)\right]^{-1}-I$$
$$=\left[\left(I+\lambda_{Algn/Scal}\right)^{-1}\left(I+\lambda_{Algn/Scal}-\lambda_{Algn/Scal}+\hat{\lambda}_{Algn/Scal}\right)\right]^{-1} \qquad \text{(A-11)}$$
$$=\left[I+\left(I+\lambda_{Algn/Scal}\right)^{-1}\left(\hat{\lambda}_{Algn/Scal}-\lambda_{Algn/Scal}\right)\right]^{-1}-I$$

With (A-11), (A-10) becomes the alternate form:

$$\delta\hat{\underline{a}}_{SF}^{B}=\left\{\left[I+\left(I+\lambda_{Algn/Scal}\right)^{-1}\left(\hat{\lambda}_{Algn/Scal}-\lambda_{Algn/Scal}\right)\right]^{-1}-I\right\}\underline{a}_{SF}^{B}$$
$$+\left(I+\hat{\lambda}_{Algn/Scal}\right)^{-1}\left(\underline{\lambda}_{Bias}-\hat{\underline{\lambda}}_{Bias}+\underline{\lambda}_{Quant}+\underline{\lambda}_{Rndm}\right) \qquad \text{(A-12)}$$

Equation (A-12) can be simplified by introducing definitions for $\delta\hat{\lambda}_{Algn/Scal}$, $\delta\hat{\underline{\lambda}}_{Bias}$ errors in $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$:

$$\delta\hat{\lambda}_{Algn/Scal}\equiv\hat{\lambda}_{Algn/Scal}-\lambda_{Algn/Scal} \qquad \delta\hat{\underline{\lambda}}_{Bias}\equiv\hat{\underline{\lambda}}_{Bias}-\underline{\lambda}_{Bias} \qquad \text{(A-13)}$$

With (A-13), (A-12) simplifies to

$$\delta\hat{\underline{a}}_{SF}^{B}=\left\{\left[I+\left(I+\lambda_{Algn/Scal}\right)^{-1}\delta\hat{\lambda}_{Algn/Scal}\right]^{-1}-I\right\}\underline{a}_{SF}^{B}$$
$$+\left(I+\hat{\lambda}_{Algn/Scal}\right)^{-1}\left(-\delta\hat{\underline{\lambda}}_{Bias}+\underline{\lambda}_{Quant}+\underline{\lambda}_{Rndm}\right) \qquad \text{(A-14)}$$

A.5 APPLICATION TO THE STRAPDOWN ROTATION TEST

The Strapdown Rotation Test (SRT) is designed to determine the $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$ terms in the (A-8) accelerometer calibration equations (and similarly for the strapdown gyros). The basic method is to generate SRT input acceleration measurements from an IMU containing the inertial sensors (gyros/accelerometers) calibrated using estimated values for the calibration coefficients. For the accelerometers, calibration would consist of (A-4) followed by (A-8). Based on inertial sensor error models (i.e., (A-10) or (A-14) for the accelerometers), the SRT processes the IMU output measurements to ascertain residual sensor compensation errors (e.g., in $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$ for the accelerometers). The results are then used to update the compensation term coefficients. Two cases can be considered for this process; 1) When

$\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$ are completely unknown, and 2) When $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$ are approximately known. (Appendix B considers the same cases for the gyros. The remainder of this appendix considers only the accelerometer calibration portion.)

Case 1 corresponds to a basic application of an SRT to initially determine values for $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$. The method is to apply the SRT to an IMU with acceleration measurements for the SRT calculated using (A-8) for accelerometer compensation, but with $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$ completely unknown (i.e., to be determined by the SRT). Then the error in the (A-8) output would be given by (A-10) with $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$ set to zero:

$$\delta\hat{\underline{a}}_{SF}^{B} = \lambda_{Algn/Scal}\, \underline{a}_{SF}^{B} + \underline{\lambda}_{Bias} + \underline{\lambda}_{Quant} + \underline{\lambda}_{Rndm} \qquad \text{(A-15)}$$

SRT processing of the IMU outputs then determines estimates for $\lambda_{Algn/Scal}$ and $\underline{\lambda}_{Bias}$ based on (A-15) representing the error in (A-8) accelerometer compensation used in the IMU to generate SRT input measurements.

Having completed a basic SRT Case 1 type determination of $\lambda_{Algn/Scal}$ and $\underline{\lambda}_{Bias}$ estimates, Case 2 corresponds to a second application of the SRT for Case 1 determination accuracy enhancement. Then $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$ terms used in (A-8) would be those determined from the first SRT application (containing (A-13) defined residual $\delta\hat{\lambda}_{Algn/Scal}$, $\delta\hat{\underline{\lambda}}_{Bias}$ errors), and (A-14) would represent the resulting (A-8) compensation output error. Part 1 [3, Section 5.0] shows that the magnitudes of $\delta\hat{\lambda}_{Algn/Scal}$, $\delta\hat{\underline{\lambda}}_{Bias}$ are second order relative to $\lambda_{Algn/Scal}$, $\underline{\lambda}_{Bias}$ (e.g., for $\lambda_{Algn/Scal}$, $\underline{\lambda}_{Bias}$ on the order of 1 milli-rad or 1 milli-g, $\delta\hat{\lambda}_{Algn/Scal}$, $\delta\hat{\underline{\lambda}}_{Bias}$ will be on the order of 1 micro-rad or 1 micro-g). Thus, to second order accuracy, (A-14) approximates as

$$\begin{aligned}\delta\hat{\underline{a}}_{SF}^{B} &\approx \left[\left(I + \delta\hat{\lambda}_{Algn/Scal}\right)^{-1} - I\right] \underline{a}_{SF}^{B} - \delta\hat{\underline{\lambda}}_{Bias} + \underline{\lambda}_{Quant} + \underline{\lambda}_{Rndm} \\ &\approx -\delta\hat{\lambda}_{Algn/Scal}\, \underline{a}_{SF}^{B} - \delta\hat{\underline{\lambda}}_{Bias} + \underline{\lambda}_{Quant} + \underline{\lambda}_{Rndm}\end{aligned} \qquad \text{(A-16)}$$

The SRT is structured to measure the errors in $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$ of (A-8) based on the (A-15) form for the errors. Note, however, that (A-15) and (A-16) are of exactly the same form (with $\hat{\lambda}_{Algn/Scal}$, $\hat{\underline{\lambda}}_{Bias}$ replaced by $-\delta\hat{\lambda}_{Algn/Scal}$, $-\delta\hat{\lambda}_{Algn/Scal}$). Thus, the SRT designed to determine $\hat{\lambda}_{Algn/Scal}$ and $\hat{\underline{\lambda}}_{Bias}$ errors based on the (A-15) model can also be used to determine the $-\delta\hat{\lambda}_{Algn/Scal}$, $-\delta\hat{\underline{\lambda}}_{Bias}$ residuals in (A-16). Applying the SRT to an IMU that has been (A-8) calibrated using Case 1 measured $\hat{\lambda}_{Algn/Scal}$, $\underline{\hat{\lambda}}_{Bias}$ coefficients, will then enable determination of the residual $-\delta\hat{\lambda}_{Algn/Scal}$, $-\delta\underline{\hat{\lambda}}_{Bias}$ errors in $\hat{\lambda}_{Algn/Scal}$ and $\underline{\hat{\lambda}}_{Bias}$, limited by uncertainty in the SRT rotation fixture accuracy and $\underline{\lambda}_{Quant} + \underline{\lambda}_{Rndm}$ uncertainties remaining in (A-16). Adding the Case 2 SRT determined $-\delta\hat{\lambda}_{Algn/Scal}$, $-\delta\underline{\hat{\lambda}}_{Bias}$ coefficient errors to the Case 1 calibrated $\hat{\lambda}_{Algn/Scal}$, $\underline{\hat{\lambda}}_{Bias}$ coefficients will complete the recalibration process.

A.6 APPLICATION TO SRT PROCESS DESIGN

The SRT processing equations are based on a deterministic version of Case 1 error model (A-15) in the Section A.5 two-stage compensation process, with the $\lambda_{Algn/Scal}$ term expanded as

$$\begin{aligned} \lambda_{Algn/Scal} &= \lambda_{Mis} + \lambda_{LinScal} + \lambda_{NonLinScal} + \lambda_{Asym}\, A^B_{SFSign} \\ &\approx \lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym}\, A^B_{SFSign} \end{aligned} \qquad \text{(A-17)}$$

where

λ_{Mis} = Accelerometer triad misalignment compensation error residual matrix having zero diagonal elements.

$\lambda_{LinScal}$ = Accelerometer triad linear scale factor compensation error residual diagonal matrix.

$\lambda_{NonLinScal}$ = Accelerometer triad non-linear scale factor compensation error residual diagonal matrix.

λ_{Asym} = Accelerometer triad asymmetric scale factor compensation error residual diagonal matrix.

A^B_{SFSign} = Diagonal matrix having elements of unity magnitude and sign (plus or minus) of the $\underline{a}_{SF}$ element signs (plus or minus).

Note in (A-17) that $\lambda_{NonLinScal}$ has been neglected based on the assumption that non-linear scale factor error has been adequately modeled in the (A-4) first-stage compensation equations, leaving negligible residual error. Substituting (A-17) into (A-15) and deleting the $\underline{\lambda}_{Quant}$, $\underline{\lambda}_{Rndm}$ as indeterminate then obtains the compensated accelerometer model for SRT processing equation design:

$$\delta \hat{\underline{a}}_{SF}^{B} = \left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym} \, A_{SFSign}^{B} \right) \underline{a}_{SF}^{B} + \underline{\lambda}_{Bias} \tag{A-18}$$

A.7 APPLICATION TO SRT PROCESS ERROR ANALYSIS

For SRT process error analysis, (A-15) with (A-17) (neglecting $\lambda_{NonLinScal}$) is used as the accelerometer error model during stationary measurement periods at the start and end of each SRT rotation sequence:

$$\delta \hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}} = \left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym} \, A_{SFSign}^{B_{Strt}} \right) \underline{a}_{SF_{Strt}}^{B_{Strt}} + \underline{\lambda}_{Bias} + \underline{\lambda}_{Quant_{Strt}} + \underline{\lambda}_{Rndm_{Strt}} \tag{A-19}$$

$$\begin{aligned} \delta \hat{\underline{a}}_{SF_{End}}^{B_{End}} &= \left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym} \, A_{SFSign}^{B_{End}} \right) \underline{a}_{SF_{End}}^{B_{End}} + \underline{\lambda}_{Bias} + \underline{\lambda}_{Quant_{End}} + \underline{\lambda}_{Rndm_{End}} \\ &= \left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym} \, A_{SFSign}^{B_{End}} \right) C_{B_{Strt}}^{B_{End}} \, \underline{a}_{SF_{End}}^{B_{Strt}} + \underline{\lambda}_{Bias} + \underline{\lambda}_{Quant_{End}} + \underline{\lambda}_{Rndm_{End}} \end{aligned} \tag{A-20}$$

where

$Strt$, End = Subscripts indicating parameter values at the start and end of an SRT rotation sequence.

$\delta \hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}}$ = $\delta \hat{\underline{a}}_{SF}^{B}$ during the stationary measurement period before initiating an SRT rotation sequence, in B_{Strt} coordinates (superscript).

$\delta \hat{\underline{a}}_{SF_{End}}^{B_{End}}$ = $\delta \hat{\underline{a}}_{SF}^{B}$ during the stationary measurement period after an SRT rotation sequence completion, in B_{End} coordinates (superscript).

Eq. (33) shows that the $\underline{a}_{SF_{Strt}}^{B_{Strt}}$, $\underline{a}_{SF_{End}}^{B_{Strt}}$ terms in (A-19) – (A-20) satisfy

$$\underline{a}_{SF_{End}}^{B_{Strt}} = \underline{a}_{SF_{Strt}}^{B_{Strt}} = -\, g \left(\underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}} \right) \tag{A-21}$$

where

$\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}$ = B_{Strt} frame angular error relative to its nominal orientation (B_{Strt}^{Nom}).

$\underline{u}_{Dwn}^{B_{Strt}^{Nom}}$ = Unit vector downward in nominal B_{Strt} coordinates (B_{Strt}^{Nom}).

Then with (A-21), (A-19) becomes for $\delta \hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}}$:

$$\delta\underline{\hat{a}}_{SF\,Strt}^{B\,Strt} = -\,g\left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym}\,A_{SFSign}^{B\,Strt}\right)\left(\underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right)$$

$$+\,\underline{\lambda}_{Bias} + \underline{\lambda}_{Quant\,_{Strt}} + \underline{\lambda}_{Rndm\,Strt}$$

$$= -\,g\left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym}\,A_{SFSign}^{B\,Strt}\right)\underline{u}_{Dwn}^{B_{Strt}^{Nom}} + \underline{\lambda}_{Bias} \tag{A-22}$$

$$+\,g\left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym}\,A_{SFSign}^{B\,End}\right)\left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) + \underline{\lambda}_{Quant\,_{Strt}} + \underline{\lambda}_{Rndm\,Strt}$$

Substituting for $C_{B\,Strt}^{B\,End}$ from the transpose of (30) and $\underline{a}_{SF\,End}^{B\,Strt}$ from (A-21), the $C_{B\,Strt}^{B\,End}\,\underline{a}_{SF\,End}^{B\,Strt}$ term in (A-20) becomes

$$C_{B\,Strt}^{B\,End}\,\underline{a}_{SF\,End}^{B\,Strt} = -g\,C_{B_{Strt}^{Nom}}^{B_{End}^{Nom}}\left[I - \left[\left(\underline{\alpha}_{End}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right)\times\right]\right]\left(\underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right)$$

$$\approx -g\,C_{B_{Strt}^{Nom}}^{B_{End}^{Nom}}\,\underline{u}_{Dwn}^{NED} + g\,C_{B_{Strt}^{Nom}}^{B_{End}^{Nom}}\left\{\left[\left(\underline{\alpha}_{End}^{B_{Strt}^{Nom}} - \underline{\alpha}_{Strt}^{B_{Strt}^{Nom}}\right)\times\underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right] + \left[\left(\underline{\alpha}_{Strt}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right)\times\right]\right\} \tag{A-23}$$

$$= -g\,C_{B_{Strt}^{Nom}}^{B_{End}^{Nom}}\left(\underline{u}_{Dwn}^{B_{Strt}^{Nom}} - \underline{\alpha}_{End}^{B_{Strt}^{Nom}} \times \underline{u}_{Dwn}^{B_{Strt}^{Nom}}\right) = -g\left(\underline{u}_{Dwn}^{B_{End}^{Nom}} - \underline{\alpha}_{End}^{B_{End}^{Nom}} \times \underline{u}_{Dwn}^{B_{End}^{Nom}}\right)$$

With (A-23), (A-20) then becomes for $\delta\underline{\hat{a}}_{SF\,End}^{B\,End}$:

$$\delta\underline{\hat{a}}_{SF\,End}^{B\,End} = -\,g\left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym}\,A_{SFSign}^{B\,End}\right)\left(\underline{u}_{Dwn}^{B_{End}^{Nom}} - \underline{\alpha}_{End}^{B_{End}^{Nom}} \times \underline{u}_{Dwn}^{B_{End}^{Nom}}\right)$$

$$+\,\underline{\lambda}_{Bias} + \underline{\lambda}_{Quant\,_{End}} + \underline{\lambda}_{Rndm\,End}$$

$$= -\,g\left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym}\,A_{SFSign}^{B\,End}\right)\underline{u}_{Dwn}^{B_{End}^{Nom}} + \underline{\lambda}_{Bias} \tag{A-24}$$

$$+\,g\left(\lambda_{Mis} + \lambda_{LinScal} + \lambda_{Asym}\,A_{SFSign}^{B\,End}\right)\left(\underline{\alpha}_{End}^{B_{End}^{Nom}} \times \underline{u}_{Dwn}^{B_{End}^{Nom}}\right) + \underline{\lambda}_{Quant\,_{End}} + \underline{\lambda}_{Rndm\,End}$$

Equations (A-22) and (A-24) are the analytical forms used for SRT process error analysis.

APPENDIX B

STRAPDOWN GYRO TRIAD ANALYTICAL MODELS

Analytical models are developed in this appendix describing gyro triad uncompensated outputs in terms of inertial angular rate input, gyro triad compensation equations, and associated error models.

B.1 GYRO TRIAD OUTPUT MODELS

The output vector from an uncompensated strapdown gyro triad can be characterized as a function of its inertial angular rate input vector by

$$\underline{\omega}_{Raw} = \left(I + F_{Scal} \right) \left(F_{Algn}\, \underline{\omega} + \delta\underline{\omega}_{Bias} + \delta\underline{\omega}_{Quant} + \delta\underline{\omega}_{Rndm} \right) \qquad \text{(B-1)}$$

where

$\underline{\omega}_{Raw}$ = Gyro triad uncompensated inertial angular rate output vector.

$\underline{\omega}$ = Gyro triad inertial angular rate input vector.

I = Identity matrix.

F_{Scal} = Gyro triad scale factor diagonal error matrix. Nominally, the F_{Scal} matrix is zero. The F_{Scal} matrix may include non-linear scale factor effects and temperature dependency.

F_{Algn} = Gyro triad alignment matrix. Nominally, the F_{Algn} matrix is identity. The F_{Algn} matrix may include temperature dependency.

$\delta\underline{\omega}_{Bias}$ = Gyro triad bias vector. Each element equals the systematic output from a particular gyro under zero inertial angular rate input conditions. For some gyros, $\delta\underline{\omega}_{Bias}$ may have temperature and specific force acceleration sensitivities.

$\delta\underline{\omega}_{Quant}$ = Gyro triad pulse quantization error vector caused by gyro outputs per axis only provided when the cumulative input equals the gyro pulse weight.

$\delta\underline{\omega}_{Rndm}$ = Gyro triad random output error vector.

An alternative form of the uncompensated gyro triad output model derives from (B-1) as follows:

$$
\begin{aligned}
\underline{\omega}_{Raw} &= (I + F_{Scal})\left(F_{Algn}\ \underline{\omega} + \delta\underline{\omega}_{Bias} + \delta\underline{\omega}_{Quant} + \delta\underline{\omega}_{Rndm}\right) \\
&= (I + F_{Scal}) F_{Algn}\ \underline{\omega} + (I + F_{Scal})\left(\delta\underline{\omega}_{Bias} + \delta\underline{\omega}_{Quant} + \delta\underline{\omega}_{Rndm}\right) \\
&= \underline{\omega} + \left[(I + F_{Scal}) F_{Algn} - I\right]\underline{\omega} \\
&\quad + (I + F_{Scal})\left(\delta\underline{\omega}_{Bias} + \delta\underline{\omega}_{Quant} + \delta\underline{\omega}_{Rndm}\right) \\
&= \underline{\omega} + \kappa_{Algn/Scal}\ \underline{\omega} + \underline{\kappa}_{Bias} + \underline{\kappa}_{Quant} + \underline{\kappa}_{Rndm}
\end{aligned} \tag{B-2}
$$

in which

$$
\begin{aligned}
&\kappa_{Algn/Scal} \equiv (I + F_{Scal}) G_{Algn} - I \qquad \underline{\kappa}_{Bias} \equiv (I + F_{Scal})\delta\underline{\omega}_{Bias} \\
&\underline{\kappa}_{Quant} \equiv (I + F_{Scal})\delta\underline{\omega}_{Quant} \qquad \underline{\kappa}_{Rndm} \equiv (I + F_{Scal})\delta\underline{\omega}_{Rndm}
\end{aligned} \tag{B-3}
$$

B.2 GYRO TRIAD COMPENSATION ALGORITHMS

Compensation formulas to correct the (B-1) basic model output are structured based on the inverse of (B-1) with random noise terms deleted:

$$
\underline{\omega}' = (I + K_{Scal})^{-1}\ \underline{\omega}_{Raw} \qquad \underline{\tilde{\omega}} = K_{Algn}^{-1}\left(\underline{\omega}' - \underline{\omega}_{Bias} - \underline{\omega}_{Quant}\right) \tag{B-4}
$$

where

$\underline{\tilde{\omega}}$ = Compensated gyro triad output vector.

K_{Scal} = Gyro triad scale factor correction matrix, a diagonal matrix in which each element adjusts the output scaling to correspond to the actual scaling for the particular sensor output. Typically modeled by analytical equations containing premeasured coefficients. Nominally, the K_{Scal} matrix is zero. The K_{Scal} matrix may include non-linear scale factor effects and temperature dependency.

K_{Algn} = Gyro triad alignment correction matrix, typically modeled by a set of equations with premeasured coefficients. Nominally, the K_{Algn} matrix is identity. The K_{Algn} matrix may include temperature dependency.

$\underline{\omega}_{Bias}$ = Gyro triad bias correction vector with components typically modeled by analytical equations containing premeasured coefficients. Each element corrects the output of a particular gyro to zero under zero input inertial angular rate conditions. In some gyros, $\underline{\omega}_{Bias}$ may have temperature and specific force acceleration sensitivities.

$\underline{\omega}_{Quant}$ = Gyro triad pulse quantization correction vector for gyro outputs only being provided when the cumulative input equals the pulse weight per axis. Includes pulse output logic dead-band effect under turn-around conditions (See [2, Sect. 8.1.3.2]).

Similar to (B-4), a compensation formula can also be structured to correct the (B-2) alternative gyro model output as the inverse of (B-2) with the random errors excluded:

$$\hat{\underline{\omega}} = \left(I + \kappa_{Algn/Scal}\right)^{-1} \left(\underline{\omega}_{Raw} - \underline{\kappa}_{Bias} - \underline{\kappa}_{Quant}\right) \qquad \text{(B-5)}$$

where

$\hat{\underline{\omega}}$ = Compensated alternative gyro triad model output.

B.3 APPLICATION TO A TWO-STAGE COMPENSATION STRUCTURE

In many applications, a two-stage compensation structure is incorporated whereby (B-4) is used as the first stage compensation to correct gyro outputs followed by (B-5) to correct residual errors in the (B-4) compensation. With such an approach, the $\underline{\omega}_{Raw}$ input in (B-5) would represent the (B-4) output. Using this method, it is also assumed that the (B-4) compensation leaves residual errors in the (B-4) output due to inaccuracy in determining the (B-4) compensation correction terms (K_{Algn}, K_{Scal}, $\underline{\omega}_{Bias}$, $\underline{\omega}_{Quant}$) and gyro changes since determination.

Note: The K_{Algn}, K_{Scal}, $\underline{\omega}_{Bias}$ term coefficients are typically measured for each individual sensor prior to installation in the IMU sensor assembly, and $\underline{\omega}_{Quant}$ may (or may not) be only approximately modeled as a design characteristic. Note also that the K_{Algn} matrix can only be accurately determined after installation in an IMU due to mounting uncertainties.

For a two-stage compensation structure, the alternative (B-2) model and its (B-5) compensation equation with coordinate frame designation become:

$$\tilde{\underline{\omega}}^B = \underline{\omega}^B + \kappa_{Algn/Scal}\, \underline{\omega}^B + \underline{\kappa}_{Bias} + \underline{\kappa}_{Quant} + \underline{\kappa}_{Rndm} \qquad \text{(B-6)}$$

$$\hat{\underline{\omega}}^B = \left(I + \kappa_{Algn/Scal}\right)^{-1} \left(\tilde{\underline{\omega}}^B - \underline{\kappa}_{Bias}\right) \qquad \text{(B-7)}$$

where

$\underline{\tilde{\omega}}^B$ = Inertial angular rate output vector from the first stage (B-4) compensation equations in B frame coordinates, and containing compensation error residuals ($\tilde{}$ designation).

$\underline{\omega}^B$ = True (error free) inertial angular rate vector in B frame coordinates.

$\underline{\hat{\omega}}^B$ = Second stage compensated gyro triad output vector (in B frame coordinates) generated by second stage compensation (B-7) using first stage compensation (B-4) output as the (B-7) input.

$\kappa_{Algn/Scal}$ = Gyro triad alignment/scale-factor compensation error residual matrix.

$\underline{\kappa}_{Bias}$ = Gyro triad bias compensation error residual vector.

$\underline{\kappa}_{Quant}$ = Gyro triad output quantization compensation error residual vector.

$\underline{\kappa}_{Rndm}$ = Gyro triad output random noise vector.

B.4 ERROR MODELS FOR THE TWO-STAGE COMPENSATED GYRO TRIAD

An error model for the two-stage compensated gyro triad output $\underline{\hat{\omega}}^B$ is defined as the (B-7) output minus the true inertial angular rate $\underline{\omega}^B$. For clarity, we first rewrite (B-7) as

$$\underline{\hat{\omega}}^B = \left(I + \hat{\kappa}_{Algn/Scal}\right)^{-1} \left(\underline{\tilde{\omega}}^B - \underline{\hat{\kappa}}_{Bias}\right) \qquad \text{(B-8)}$$

using the $\hat{}$ notation to indicate that the error coefficient terms may still contain error. Then the error in (B-8) defines as

$$\delta\underline{\hat{\omega}}^B \equiv \underline{\hat{\omega}}^B - \underline{\omega}^B = \left(I + \hat{\kappa}_{Algn/Scal}\right)^{-1} \left(\underline{\tilde{\omega}}^B - \underline{\hat{\kappa}}_{Bias}\right) - \underline{\omega}^B \qquad \text{(B-9)}$$

where

$\delta\underline{\hat{\omega}}^B$ = Error in $\underline{\hat{\omega}}^B$.

Substituting (B-6) for $\underline{\tilde{\omega}}^B$ finds for the (B-9) output error:

$$\delta\underline{\hat{\omega}}^B = \left(I + \hat{\kappa}_{Algn/Scal}\right)^{-1}\left(\underline{\tilde{\omega}}^B - \underline{\hat{\kappa}}_{Bias}\right) - \underline{\omega}^B$$

$$= \left(I + \hat{\kappa}_{Algn/Scal}\right)^{-1}\left(\underline{\omega}^B + \kappa_{Algn/Scal}\,\underline{\omega}^B + \underline{\kappa}_{Bias} + \underline{\kappa}_{Quant} + \underline{\kappa}_{Rndm} - \underline{\hat{\kappa}}_{Bias}\right) - \underline{\omega}^B \tag{B-10}$$

$$= \left[\left(I + \hat{\kappa}_{Algn/Scal}\right)^{-1}\left(I + \kappa_{Algn/Scal}\right) - I\right]\underline{\omega}^B$$

$$+ \left(I + \hat{\kappa}_{Algn/Scal}\right)^{-1}\left(\underline{\kappa}_{Bias} - \underline{\hat{\kappa}}_{Bias} + \underline{\kappa}_{Quant} + \underline{\kappa}_{Rndm}\right)$$

An alternate form of (B-10) can be derived by first writing the $\underline{\omega}^B$ coefficient in (B-10) in the equivalent form:

$$\left(I + \hat{\kappa}_{Algn/Scal}\right)^{-1}\left(I + \kappa_{Algn/Scal}\right) - I = \left[\left(I + \kappa_{Algn/Scal}\right)^{-1}\left(I + \hat{\kappa}_{Algn/Scal}\right)\right]^{-1} - I$$

$$= \left[\left(I + \kappa_{Algn/Scal}\right)^{-1}\left(I + \kappa_{Algn/Scal} - \kappa_{Algn/Scal} + \hat{\kappa}_{Algn/Scal}\right)\right]^{-1} \tag{B-11}$$

$$= \left[I + \left(I + \kappa_{Algn/Scal}\right)^{-1}\left(\hat{\kappa}_{Algn/Scal} - \kappa_{Algn/Scal}\right)\right]^{-1} - I$$

With (B-11), (B-10) becomes the alternate form:

$$\delta\underline{\hat{\omega}}^B = \left\{\left[I + \left(I + \kappa_{Algn/Scal}\right)^{-1}\left(\hat{\kappa}_{Algn/Scal} - \kappa_{Algn/Scal}\right)\right]^{-1} - I\right\}\underline{\omega}^B$$
$$+ \left(I + \hat{\kappa}_{Algn/Scal}\right)^{-1}\left(\underline{\kappa}_{Bias} - \underline{\hat{\kappa}}_{Bias} + \underline{\kappa}_{Quant} + \underline{\kappa}_{Rndm}\right) \tag{B-12}$$

Equation (B-12) can be simplified by introducing definitions for $\delta\hat{\kappa}_{Algn/Scal}$, $\delta\underline{\hat{\kappa}}_{Bias}$ errors in $\hat{\kappa}_{Algn/Scal}$ and $\underline{\hat{\kappa}}_{Bias}$:

$$\delta\hat{\kappa}_{Algn/Scal} \equiv \hat{\kappa}_{Algn/Scal} - \kappa_{Algn/Scal} \qquad \delta\underline{\hat{\kappa}}_{Bias} \equiv \underline{\hat{\kappa}}_{Bias} - \underline{\kappa}_{Bias} \tag{B-13}$$

With (B-13), (B-12) simplifies to

$$\delta\underline{\hat{\omega}}^B = \left\{\left[I + \left(I + \kappa_{Algn/Scal}\right)^{-1}\delta\hat{\kappa}_{Algn/Scal}\right]^{-1} - I\right\}\underline{\omega}^B$$
$$+ \left(I + \hat{\kappa}_{Algn/Scal}\right)^{-1}\left(-\delta\underline{\hat{\kappa}}_{Bias} + \underline{\kappa}_{Quant} + \underline{\kappa}_{Rndm}\right) \tag{B-14}$$

B.5 APPLICATION TO THE STRAPDOWN ROTATION TEST

The Strapdown Rotation Test (SRT) is designed to determine the $\hat{\kappa}_{Algn/Scal}$ term in the (B-8) gyro calibration equations (and similarly for the strapdown accelerometers. For the accelerometers, the $\underline{\hat{\lambda}}_{Bias}$ error is also included in the determination process. For the SRT gyro error determination process, the $\underline{\hat{\kappa}}_{Bias}$ error in (B-8) is considered negligible.) The basic method is to generate SRT input acceleration measurements from an IMU containing the inertial sensors (gyros/accelerometers) calibrated using estimated values for the calibration coefficients. For the gyros, calibration would consist of (B-4) followed by (B-8). Based on inertial sensor error models (i.e., (B-10) or (B-14) for the gyros), the SRT processes the IMU output measurements to ascertain residual sensor compensation errors (in $\hat{\kappa}_{Algn/Scal}$ for the gyros). The results are then used to update the compensation term coefficients. Two cases can be considered for this process; 1) When $\hat{\kappa}_{Algn/Scal}$ is completely unknown, and 2) When $\hat{\kappa}_{Algn/Scal}$ is approximately known. (Appendix A considers the same cases for the accelerometers. The remainder of this appendix considers only the gyro calibration portion.)

Case 1 corresponds to a basic application of the SRT to initially determine the value for $\hat{\kappa}_{Algn/Scal}$. The method is to apply the SRT to an IMU with acceleration measurements for the SRT calculated using (B-8) for gyro compensation, but with $\hat{\kappa}_{Algn/Scal}$ completely unknown (i.e., $\hat{\kappa}_{Algn/Scal}$ to be determined by the SRT). Then the error in the (B-8) output would be given by (B-10) with $\hat{\kappa}_{Algn/Scal}$ set to zero (and $\underline{\hat{\kappa}}_{Bias}$ set to zero or to a value determined by a different method prior to SRT execution):

$$\delta\underline{\hat{\omega}}^B = \kappa_{Algn/Scal}\,\underline{\omega}^B - \delta\underline{\hat{\kappa}}_{Bias} + \underline{\kappa}_{Quant} + \underline{\kappa}_{Rndm} \qquad \text{(B-15)}$$

in which $\delta\underline{\hat{\kappa}}_{Bias}$ is as defined previously in (B-13). SRT processing of the IMU outputs then determines an estimate for $\kappa_{Algn/Scal}$ based on (B-15) representing the error in (B-8) gyro compensation used in the IMU to generate SRT input measurements, and also assuming that $\delta\underline{\hat{\kappa}}_{Bias}$ is negligibly small.

Having completed a basic SRT Case 1 type determination of the $\kappa_{Algn/Scal}$ estimate, Case 2 corresponds to a second application of the SRT for Case 1 determination accuracy enhancement. Then $\hat{\kappa}_{Algn/Scal}$ in (B-8) would be the value determined from the first SRT application (containing (B-13) defined residual $\delta\hat{\kappa}_{Algn/Scal}$ error), $\underline{\hat{\kappa}}_{Bias}$ would be set to zero or to value determined by a different method prior to SRT execution, and (B-14) would represent the resulting (B-8) compensation output error. Part 1 [3, Section 5.0] shows that the magnitude of $\delta\hat{\kappa}_{Algn/Scal}$ is second order relative to $\kappa_{Algn/Scal}$ (e.g., for $\kappa_{Algn/Scal}$ on the order of 1 milli-rad, $\delta\hat{\kappa}_{Algn/Scal}$, will be on the order of 1 micro-rad). Thus, to second order accuracy, (B-14) approximates as

$$\begin{aligned}\delta\underline{\hat{\omega}}^B &\approx \left[\left(I+\delta\hat{\kappa}_{Algn/Scal}\right)^{-1}-I\right]\underline{\omega}^B-\delta\underline{\hat{\kappa}}_{Bias}+\underline{\kappa}_{Quant}+\underline{\kappa}_{Rndm} \\ &\approx -\delta\hat{\kappa}_{Algn/Scal}\,\underline{\omega}^B-\delta\underline{\hat{\kappa}}_{Bias}+\underline{\kappa}_{Quant}+\underline{\kappa}_{Rndm}\end{aligned} \qquad \text{(B-16)}$$

The SRT is structured to measure the error in $\hat{\kappa}_{Algn/Scal}$ of (B-8) based on the (B-15) form for the error. Note, however, that (B-15) and (B-16) are of exactly the same form (with $\hat{\kappa}_{Algn/Scal}$ replaced by $-\delta\hat{\kappa}_{Algn/Scal}$). Thus, the SRT designed to determine $\hat{\kappa}_{Algn/Scal}$ error based on the (B-15) model can also be used to determine the $-\delta\hat{\kappa}_{Algn/Scal}$ residual in (B-16). Applying the SRT to an IMU that has been (B-8) calibrated using Case 1 determined $\hat{\kappa}_{Algn/Scal}$ coefficients, will then enable determination of the residual $-\delta\hat{\kappa}_{Algn/Scal}$ error in $\hat{\kappa}_{Algn/Scal}$, limited by uncertainty in the SRT rotation fixture accuracy and $-\delta\underline{\hat{\kappa}}_{Bias}+\underline{\kappa}_{Quant}+\underline{\kappa}_{Rndm}$ uncertainties remaining in (B-16). Adding the Case 2 SRT determined $-\delta\hat{\kappa}_{Algn/Scal}$ coefficient errors to the Case 1 calibrated $\hat{\kappa}_{Algn/Scal}$ coefficients will complete the recalibration process.

B.6 APPLICATION TO SRT PROCESS DESIGN

The SRT processing equations are based on a deterministic version of Case 1 error model (B-15) in the Section B.5 two-stage compensation process, with the $\kappa_{Algn/Scal}$ term expanded as

$$\begin{aligned}\kappa_{Algn/Scal} &= \kappa_{Mis}+\kappa_{LinScal}+\kappa_{NonLinScal}+\kappa_{Asym}\,\Omega^B_{Sign} \\ &\approx \kappa_{Mis}+\kappa_{LinScal}+\kappa_{Asym}\,\Omega^B_{Sign}\end{aligned} \qquad \text{(B-17)}$$

where

κ_{Mis} = Gyro triad misalignment compensation error residual matrix having zero diagonal elements.

$\kappa_{LinScal}$ = Gyro triad linear scale factor compensation error residual diagonal matrix.

$\kappa_{NonLinScal}$ = Gyro triad non-linear scale factor compensation error residual diagonal matrix.

κ_{Asym} = Gyro triad asymmetric scale factor compensation error residual diagonal matrix.

Ω_{Sign}^{B} = Diagonal matrix having elements of unity magnitude and sign (plus or minus) of the $\underline{\omega}^{B}$ element signs (plus or minus).

Note in (B-17) that $\kappa_{NonLinScal}$ has been neglected based on the assumption that non-linear scale factor error has been adequately modeled in the (B-4) first-stage compensation equations, leaving negligible residual error. Substituting (B-17) into (B-15), deleting the $\underline{\kappa}_{Quant}$, $\underline{\kappa}_{Rndm}$ as indeterminate, and deleting $\delta\hat{\underline{\kappa}}_{Bias}$ as negligible, then obtains the compensated gyro model for SRT processing equation design:

$$\hat{\underline{\omega}}^{B} = \left(\kappa_{Mis} + \kappa_{LinScal} + \kappa_{Asym}\Omega_{Sign}^{B}\right)\underline{\omega}^{B} \qquad \text{(B-18)}$$

B.7 APPLICATION TO SRT PROCESS ERROR ANALYSIS

For SRT process error analysis, (B-15) with (B-17) (neglecting $\kappa_{NonLinScal}$) is used as the gyro error model during each SRT rotation sequence. With more specificity for $\underline{\omega}^{B}$, the result is

$$\delta\hat{\underline{\omega}}_{I:B}^{B} = \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\,\Omega_{IBSign}^{B}\right)\underline{\omega}_{I:B}^{B} - \delta\hat{\underline{\kappa}}_{Bias} + \underline{\kappa}_{Quant} + \underline{\kappa}_{Rndm} \qquad \text{(B-19)}$$

where

$\underline{\omega}_{I:B}^{B}$ = True (error free) angular rate of the B frame relative to non-rotating inertial space ($I:B$ subscript) in B frame coordinates (superscript).

Ω_{IBSign}^{B} = Diagonal matrix having elements of unity magnitude and sign (plus or minus) of the $\underline{\omega}_{I:B}^{B}$ element signs (plus or minus).

$\delta\hat{\underline{\omega}}_{I:B}^{B}$ = Error in the $\hat{\underline{\omega}}_{I:B}^{B}$ output from (B-8), i.e., the error in the compensation estimate for $\underline{\omega}_{I:B}^{B}$.

The $\underline{\omega}_{I:B}^{B}$ term in (B-19) can be expanded as

$$
\begin{aligned}
\underline{\omega}_{I:B}^{B} &= \underline{\omega}_{I:E}^{B} + \underline{\omega}_{E:B}^{B} = C_{B^{Nom}}^{B}\left(\underline{\omega}_{I:E}^{B^{Nom}} + \underline{\omega}_{E:B}^{B^{Nom}}\right) \\
&= C_{B^{Nom}}^{B}\left(\underline{\omega}_{I:E}^{B^{Nom}} + \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} + \underline{\omega}_{B^{Nom}:B}^{B^{Nom}}\right)
\end{aligned}
\tag{B-20}
$$

where

$\underline{\omega}_{I:E}^{B}$, $\underline{\omega}_{I:E}^{B^{Nom}}$ = Earth rotation rate relative to inertial space ($I:E$ subscript) in B and B^{Nom} frame coordinates (superscripts).

$\underline{\omega}_{E:B}^{B^{Nom}}$ = B frame rotation rate relative to the earth ($E:B$ subscript) in B^{Nom} frame coordinates (superscript).

$\underline{\omega}_{E:B^{Nom}}^{B^{Nom}}$ = B^{Nom} frame rotation rate relative to the earth ($E:B^{Nom}$ subscript) in B^{Nom} frame coordinates (superscript).

$\underline{\omega}_{B^{Nom}:B}^{B^{Nom}}$ = B frame rotation rate relative to the B^{Nom} frame ($B^{Nom}:B$ subscript) in B^{Nom} frame coordinates (superscript).

Based on (27) and (29), the $C_{B^{Nom}}^{B}$ term in (B-20) approximates as

$$
C_{B^{Nom}}^{B} \approx I - \left(\underline{\alpha}^{B^{Nom}} \times\right) \tag{B-21}
$$

where

$\underline{\alpha}^{B^{Nom}}$ = Small rotation vector error within $C_{B^{Nom}}^{B}$ in B^{Nom} frame coordinates (superscript).

Based on Laning rotation vector theory [2, Ref. 15a], we can also write

$$
\dot{\underline{\alpha}}^{B^{Nom}} = \underline{\omega}_{B^{Nom}:B}^{B^{Nom}} + \frac{1}{2}\dot{\underline{\alpha}}^{B^{Nom}} \times \underline{\omega}_{B^{Nom}:B}^{B^{Nom}} + \cdots \quad \therefore \quad \underline{\omega}_{B^{Nom}:B}^{B^{Nom}} \approx \dot{\underline{\alpha}}^{B^{Nom}} \tag{B-22}
$$

Substituting (B-20) with (B-21) and (B-22) into (B-19) then finds for $\delta\hat{\underline{\omega}}_{I:B}^{B}$:

$$
\begin{aligned}
\delta\hat{\underline{\omega}}_{I:B}^{B} &= \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\,\Omega_{IBSign}^{B}\right)\underline{\omega}_{I:B}^{B} + \underline{\kappa}_{Bias} + \delta\underline{\omega}_{Quant} + \delta\underline{\omega}_{Rndm} \\
&= \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\,\Omega_{IBSign}^{B}\right) C_{B^{Nom}}^{B}\left(\underline{\omega}_{I:E}^{B^{Nom}} + \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} + \underline{\omega}_{B^{Nom}:B}^{B^{Nom}}\right) \\
&\qquad + \underline{\kappa}_{Bias} + \delta\underline{\omega}_{Quant} + \delta\underline{\omega}_{Rndm} \\
&= \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\,\Omega_{IBSign}^{B}\right)\left[I - \left(\underline{\alpha}^{B^{Nom}}\times\right)\right]\left(\underline{\omega}_{I:E}^{B^{Nom}} + \underline{\omega}_{E:B^{Nom}}^{B^{Nom}} + \dot{\underline{\alpha}}^{B^{Nom}}\right) \\
&\qquad + \underline{\kappa}_{Bias} + \delta\underline{\omega}_{Quant} + \delta\underline{\omega}_{Rndm} \\
&\approx \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\,\Omega_{IBSign}^{B}\right)\left(\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} + \underline{\omega}_{I:E}^{B^{Nom}}\right) \\
&\quad + \left(\kappa_{LinScal} + \kappa_{Mis} + \kappa_{Asym}\,\Omega_{IBSign}^{B}\right)\left(-\,\underline{\alpha}^{B^{Nom}}\times\underline{\omega}_{E:B^{Nom}}^{B^{Nom}} + \dot{\underline{\alpha}}^{B^{Nom}}\right) \\
&\qquad + \underline{\kappa}_{Bias} + \delta\underline{\omega}_{Quant} + \delta\underline{\omega}_{Rndm}
\end{aligned}
\tag{B-23}
$$

Equation (B-23) is the analytical form used for SRT process error analysis.

REFERENCES

[1] Savage, P. G., “Calibration Procedures For Laser Gyro Strapdown Inertial Navigation Systems”, 9th Annual Electro-Optics / Laser Conference and Exhibition, Anaheim, California, October 25-27, 1977.

[2] Savage, P. G., *Strapdown Analytics*, Strapdown Associates, Inc., Maple Plain, Minnesota, 2000, or *Strapdown Analytics - Second Edition*, Strapdown Associates, Inc., Maple Plain, Minnesota, 2007, Section 18.4

[3] Savage, P.G., “Improved Strapdown Inertial System Calibration Procedures, Part 1, Procedures, Rotation Fixtures, And Accuracy Analysis”, WBN-14020-1, Strapdown Associates, Inc., October 20, 2017 (Updated January 11, 2018), free access available at www.strapdownassociates.com.

[4] Savage, P.G., “Improved Strapdown Inertial System Calibration Procedures, Part 3, Numerical Examples”, WBN-14020-3, Strapdown Associates, Inc., November 10, 2017, (Updated January 11, 2018), free access available at www.strapdownassociates.com.