# Modifying The Kalman Filter Measurement To Mitigate Second Order Error Amplification In INS Velocity Matching Alignment Applications

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ABSTRACT

Under benign (small horizontal velocity change) flight conditions, second order errors in traditional linearized Kalman filters can generate significant heading error in INS velocity matching initial alignment applications. The dominant cause is a second order coupling of INS heading error into the Kalman filter input measurement by the product of integrated vertical specific force (mostly 1 g upward) with the INS horizontal attitude error vector. Under benign flight conditions, this effect mimics a horizontal velocity change coupling of heading error, the primary signal normally allowing fast heading estimation under true horizontal maneuvering. This article describes a simple method for eliminating the second order error effect by modifying the form of the Kalman filter input measurement and revising the analytical definition of INS navigation coordinates.

## INTRODUCTION

An inertial navigation system (INS) calculates velocity and position by a double integration process on acceleration sensed by INS accelerometers. The direction of the accelerometer measurements is determined by INS gyros whose rotation rate outputs are processed through an integration process to continuously calculate accelerometer angular orientation (also known as "attitude"). Because the basic navigation operations in an INS (attitude, velocity, position) are integrations, proper initialization of the navigation integrators is an important part of ensuing navigation integration operations. Under dynamic flight conditions, the initialization process becomes complicated, relying on external accurate navigation inputs as a reference. INS initialization then becomes a dynamic process in which particular INS navigation parameters are compared with the reference input data, the difference providing a measure of INS initialization error, with the error measurement then fed back through appropriate gains to the INS navigation integrators so that over time, correct integrator initialization is achieved. Modern initialization processes calculate the feedback gains through a Kalman filter structure [1, 2, 3 - Chapt. 15].

Due to the high sensitivity of INS navigation integration operations to angular orientation (attitude), the critical area for initialization is reducing attitude errors (initial alignment error correction). A common structure for dynamic INS attitude alignment uses external velocity reference data compared with INS velocity as the basic measurement for feedback into the INS integrators until the measurement reaches a stable steady state [3 - Sect. 15.2.2.2]. The technique, known as "velocity matching", is based on attitude defining the accelerometer output angular orientation when being integrated into velocity/position. Consequently, attitude errors will generate errors in the INS computed velocity which, when compared with "correct" reference velocity, will over successive measurement/feedbacks, nullify the attitude errors. Simultaneously, velocity and position are updated by Kalman measurement feedback.

Prior to engaging the initialization process (also known as "Fine Alignment"), INS attitude data is initialized to approximate values (a process known as "Coarse Alignment") based on other methods depending on the application (e.g., from an Attitude Heading Reference System - AHRS, with magnetic heading feedback compensation for gyro induced heading error growth [4]). Kalman filter gains during "Fine Alignment" then correct for Coarse Alignment errors using linearized dynamic models of INS initialization and inertial sensor errors generating navigation errors into the feedback error measurement. The Coarse Alignment approach is selected so that resulting Fine Alignment initial attitude errors have negligible second order components in the linearized Kalman filter models. Reference [5], however, shows that second order errors in traditionally structured velocity matching Kalman filter measurements, while being negligible under maneuvering flight, can induce significant heading initialization error under benign flight conditions.

Reference [5] describes a method for mitigating second order errors during velocity matching by modeling them as statistical noise effects in the Kalman gain calculations. This article presents an alternative to the [5] mitigation approach using a modified form of the Fine Alignment error feedback measurement, coupled with a revised analytical definition of initial heading error. The linearized form of the resulting modified measurement for Kalman gain design is equivalent to that of the traditional measurement approach. However, noticeably absent in the modified measurement is the second order error in the traditional measurement that induces the velocity matching heading alignment error.

## NOTATION

$\underline{V}$ = Vector without specific coordinate frame designation. A vector is a parameter that has length and direction. Vectors used in the paper are classified as “free vectors”, hence, have no preferred location in coordinate frames in which they are analytically described.

$\underline{V}^A$ = Column matrix with elements equal to the projection of $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit vector parallel to that coordinate axis.

$\left(\underline{V}^{A}\times\right)$ = Skew symmetric (or cross-product) form of $\underline{V}^{A}$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which $V_{XA}, V_{YA}, V_{ZA}$ are the components of $\underline{V}^{A}$. The matrix product of $\left(\underline{V}^{A}\times\right)$ with another A frame vector equals the cross-product of $\underline{V}^{A}$ with the vector in the A frame, i.e.: $\left(\underline{V}^{A}\times\right)\underline{W}^{A} = \underline{V}^{A}\times\underline{W}^{A}$.

$C_{A_2}^{A_1}$ = Direction cosine matrix that transforms a vector from its coordinate frame $A_2$ projection form to its coordinate frame $A_1$ projection form, i.e.: $\underline{V}^{A_1} = C_{A_2}^{A_1}\,\underline{V}^{A_2}$. The columns of $C_{A_2}^{A_1}$ are projections on $A_1$ axes of unit vectors parallel to $A_2$ axes. Conversely, the rows of $C_{A_2}^{A_1}$ are projections on $A_2$ axes of unit vectors parallel to $A_1$ axes. An important property of $C_{A_2}^{A_1}$ is that it's inverse equals it's transpose.

$\underline{\omega}_{A_1A_2}$ = Angular rotation rate of coordinate frame $A_2$ relative to coordinate frame $A_1$. Conversely, the angular rotation rate of coordinate frame $A_1$ relative to coordinate frame $A_2$ is the negative of $\underline{\omega}_{A_1A_2}$, i.e.,: $\underline{\omega}_{A_2A_1} = -\underline{\omega}_{A_1A_2}$.

$\dot{(\;)} = \dfrac{d(\;)}{dt}$ = Derivative with respect to time t.

$\widehat{(\;)}$ = Computed or measured value of parameter ( ) that, in contrast with the idealized error free value ( ), contains errors.

## TRADITIONAL VELOCITY MATCHING OBSERVATION AND MEASUREMENT EQUATIONS

The traditional form of the Observation Equation for a velocity matching type Kalman aided alignment process is [3 - Sect. 15.2.2.2]:

$$\widehat{\underline{M}}_{Trd}^{N} = \widehat{\underline{v}}^{N} - \widehat{\underline{v}}_{Ref}^{N} \qquad (1)$$

where

N = INS locally level navigation coordinate frame (with Z axis up) used for attitude referencing and velocity/position integration operations.

Trd = Subscript denoting the traditional form of the associated parameter.

$\underline{v}$ = INS computed velocity vector.

$\underline{v}_{Ref}$ = Velocity vector provided by the velocity reference aiding device. For the traditional measurement approach, $\underline{v}_{Ref}$ is provided to Observation Equation (1) in N frame coordinates for comparison with INS N frame computed components of velocity $\underline{v}$.

Note - For simplicity in this article, (1) does not allow for differences between $\hat{\underline{v}}^N$ and $\hat{\underline{v}}^N_{Ref}$ that would normally be included in an actual system design (e.g., due to physical separation between the INS and reference navigation device under vehicle angular motion, i.e., so-called "lever arm" effects [3 - Sect. 15.2.2.2]). For a velocity-matching alignment configuration in which Kalman computed gains are applied continuously for INS navigation parameter feedback correction, (1) would be the feedback error measurement multiplying the gains (with allowances included to account for the difference between the measurement time and the time for gain-computation-completion/feedback-application.)

Kalman filter theory [1, 2] requires that (1) be "unbiased" (i.e., that it is only a function of unknown errors) so that the equivalent idealized error free form satisfies:

$$\underline{M}^N_{Trd} = \underline{v}^N - \underline{v}^N_{Ref} = 0 \tag{2}$$

Defining the errors in $\hat{\underline{v}}^N$ and $\hat{\underline{v}}^N_{Ref}$ as:

$$\delta\underline{v}^N \equiv \hat{\underline{v}}^N - \underline{v}^N \qquad \delta\underline{v}^N_{Ref} \equiv \hat{\underline{v}}^N_{Ref} - \underline{v}^N_{Ref} \tag{3}$$

or $\underline{v}^N = \hat{\underline{v}}^N - \delta\underline{v}^N$ and $\underline{v}^N_{Ref} = \hat{\underline{v}}^N_{Ref} - \delta\underline{v}^N_{Ref}$. Substituting in (2) then finds

$$\underline{M}^N_{Trd} = \left(\hat{\underline{v}}^N - \delta\underline{v}^N\right) - \left(\hat{\underline{v}}^N - \delta\underline{v}^N_{Ref}\right) = \hat{\underline{M}}^N_{Trd} - \delta\underline{v}^N + \delta\underline{v}^N_{Ref} = 0 \tag{4}$$

From (4), $\hat{\underline{M}}^N_{Trd}$ in (1) is equivalently the analytical function of errors:

$$\hat{\underline{M}}^N_{Trd} = \delta\underline{v}^N - \delta\underline{v}^N_{Ref} \tag{5}$$

The Measurement Equation for the traditionally used velocity matching alignment Kalman filter design is the linearized form of Observation Equation (5) which, for this simple case, is [3 - Sect. 15.2.2.2]

$$\underline{z}^N_{Trd} = \delta\underline{v}^N_{Lin} - \delta\underline{v}^N_{Ref_{Lin}} \approx \delta\underline{v}^N_{Lin} - \delta\underline{v}^N_{Ref} \tag{6}$$

where

$\underline{z}$ = Measurement for Kalman gain design.

Lin = Subscript denoting linearized form of the associated parameter (i.e., neglecting second order and higher terms).

## MODIFIED FINE ALIGNMENT VELOCITY MATCHING OBSERVATION AND MEASUREMENT EQUATIONS

The modified form of Observation Equation (1) described in this article is:

$$\hat{\underline{M}}^N_{Mod} = \hat{\underline{v}}^N - \hat{C}^N_{N*}\, \hat{\underline{v}}^{N*}_{Ref} \tag{7}$$

where

N* = Locally level navigation coordinate frame (with Z axis up) used to deliver velocity reference data $\hat{\underline{v}}^{N*}_{Ref}$ to the INS being aligned. The heading angle misalignment between the N and N* frames is the means for accounting for initial heading error in the INS attitude data at the start of alignment.

Mod = Subscript referring to the modified form of the associated parameter.

Kalman filter theory requires that (7) be an unbiased error measurement so that the equivalent error free value of (7) is:

$$\underline{M}^N_{Mod} = \underline{v}^N - C^N_{N*}\, \underline{v}^{N*}_{Ref} = 0 \tag{8}$$

Substituting for error definitions between (7) and (8) finds for (8) with (7):

$$\begin{aligned} \underline{M}^N_{Mod} &= \left(\hat{\underline{v}}^N - \delta\underline{v}^N\right) - \left(\hat{C}^N_{N*} - \delta C^N_{N*}\right)\left(\hat{\underline{v}}^{N*}_{Ref} - \delta\underline{v}^{N*}_{Ref}\right) = 0 \\ &= \hat{\underline{v}}^N - \delta\underline{v}^N - \hat{C}^N_{N*}\, \hat{\underline{v}}^{N*}_{Ref} + \hat{C}^N_{N*}\, \delta\underline{v}^{N*}_{Ref} + \delta C^N_{N*}\, \hat{\underline{v}}^{N*}_{Ref} - \delta C^N_{N*}\, \delta\underline{v}^{N*}_{Ref} \\ &= \hat{\underline{M}}^N_{Mod} - \delta\underline{v}^N + \hat{C}^N_{N*}\, \delta\underline{v}^{N*}_{Ref} + \delta C^N_{N*}\, \hat{\underline{v}}^{N*}_{Ref} - \delta C^N_{N*}\, \delta\underline{v}^{N*}_{Ref} \end{aligned} \tag{9}$$

with $\delta C^N_{N*}$ and $\delta \underline{v}^{N*}_{Ref}$, the errors in $\widehat{C}^N_{N*}$ and $\widehat{\underline{v}}^{N*}_{Ref}$, defined as

$$\delta C^N_{N*} \equiv \widehat{C}^N_{N*} - C^N_{N*} \qquad \delta \underline{v}^{N*}_{Ref} \equiv \widehat{\underline{v}}^{N*}_{Ref} - \underline{v}^{N*}_{Ref} \tag{10}$$

Then from (9) with (10),

$$\begin{aligned} \widehat{\underline{M}}^N_{Mod} &= \delta \underline{v}^N - \widehat{C}^N_{N*}\, \delta \underline{v}^{N*}_{Ref} - \delta C^N_{N*}\, \widehat{\underline{v}}^{N*}_{Ref} + \delta C^N_{N*}\, \delta \underline{v}^{N*}_{Ref} \\ &\approx \delta \underline{v}^N - \widehat{C}^N_{N*}\, \delta \underline{v}^{N*}_{Ref} - \delta C^N_{N*}\, \widehat{\underline{v}}^{N*}_{Ref} \end{aligned} \tag{11}$$

The attitude parameters in (11) are further defined by representing $C^N_{N*}$ as a heading rotation (about the vertical) from N* to N through a small constant angle $\beta$:

$$\begin{aligned} C^N_{N*} &= I + \beta \left(\underline{u}^N_Z \times\right) + \frac{1}{2}\, \beta^2 \left(\underline{u}^N_Z \times\right)\left(\underline{u}^N_Z \times\right) + \cdots \\ \widehat{C}^N_{N*} &= I + \widehat{\beta} \left(\underline{u}^N_Z \times\right) + \frac{1}{2}\, \widehat{\beta}^2 \left(\underline{u}^N_Z \times\right)\left(\underline{u}^N_Z \times\right) + \cdots \end{aligned} \tag{12}$$

$$\delta C^N_{N*} \equiv \widehat{C}^N_{N*} - C^N_{N*} = \left(\widehat{\beta} - \beta\right)\left(\underline{u}^N_Z \times\right) + \frac{1}{2}\left(\widehat{\beta}^2 - \beta^2\right)\left(\underline{u}^N_Z \times\right)\left(\underline{u}^N_Z \times\right) + \cdots$$

or

$$\delta C^N_{N*} = (1 + \widehat{\beta})\, \delta\beta \left(\underline{u}^N_Z \times\right) - \frac{1}{2}\, \delta\beta^2 \left(\underline{u}^N_Z \times\right)\left(\underline{u}^N_Z \times\right) + \cdots \tag{13}$$

where

I = Identity matrix.

$$\delta\beta \equiv \widehat{\beta} - \beta \qquad \text{or} \qquad \beta = \widehat{\beta} - \delta\beta \tag{14}$$

with, because $\beta$ is defined to be constant,

$$\dot{\beta} = 0 \qquad \dot{\widehat{\beta}} = 0 \qquad \delta\dot{\beta} = 0 \tag{15}$$

Applying (13) in (11), the Observation Equation for the modified alignment approach then becomes in terms of error parameters:

$$\begin{aligned} \widehat{\underline{M}}^N_{Mod} &= \delta \underline{v}^N - \widehat{C}^N_{N*}\, \delta \underline{v}^{N*}_{Ref} - \left[ (1 + \widehat{\beta})\, \delta\beta \left(\underline{u}^N_Z \times\right) - \frac{1}{2}\, \delta\beta^2 \left(\underline{u}^N_Z \times\right)\left(\underline{u}^N_Z \times\right) \right] \widehat{\underline{v}}^{N*}_{Ref} + \cdots \\ &= \delta \underline{v}^N - \widehat{C}^N_{N*}\, \delta \underline{v}^{N*}_{Ref} + (1 + \widehat{\beta})\, \delta\beta\, \widehat{\underline{v}}^{N*}_{Ref_H} \times \underline{u}^N_Z - \frac{1}{2}\, \delta\beta^2\, \widehat{\underline{v}}^{N*}_{Ref_H} + \cdots \end{aligned} \tag{16}$$

where

H = Subscript indicating the horizontal components of the associated vector.

For the modified approach, the measurement $\underline{z}^N_{Mod}$ for Kalman filter implementation is the linearized version of (16):

$$\underline{z}^N_{Mod} = \delta\underline{v}^N_{Lin} - \delta\underline{v}^{N*}_{Ref} + \delta\beta\ \hat{\underline{v}}^{N*}_{Ref_H} \times \underline{u}^N_Z \tag{17}$$

VELOCITY ERROR INPUTS TO THE MEASUREMENT EQUATIONS

Eqs. (1) and (6) for the traditional velocity matching alignment approach and (16) and (17) for the modified approach contain a $\delta\underline{v}^N$ velocity error term. This is obtained by integration of the $\delta\underline{v}^N$ rate equation, which from [5 - Eq. (91)], is given to second order accuracy by

$$\begin{aligned}\delta\dot{\underline{v}}^N \approx\ & \hat{C}^N_B\ \delta\underline{a}^B_{SF} + \hat{\underline{a}}^N_{SF} \times \underline{\gamma}^N + \delta\underline{g}^N_P \\ & - \left(\delta\underline{\omega}^N_{IN} + \delta\underline{\omega}^N_{IE}\right) \times \hat{\underline{v}}^N - \left(\hat{\underline{\omega}}^N_{IN} + \hat{\underline{\omega}}^N_{IE}\right) \times \delta\underline{v}^N \\ & - \frac{1}{2}\left(\hat{\underline{a}}^N_{SF} \times \underline{\gamma}^N\right) \times \underline{\gamma}^N - \left(\hat{C}^N_B\ \delta\underline{a}^B_{SF}\right) \times \underline{\gamma}^N + \left(\delta\underline{\omega}^N_{IN} + \delta\underline{\omega}^N_{IE}\right) \times \delta\underline{v}^N\end{aligned} \tag{18}$$

with

$$\delta\underline{\omega}^N_{IE} \equiv \hat{\underline{\omega}}^N_{IE} - \underline{\omega}^N_{IE} \qquad \delta\underline{\omega}^N_{IN} \equiv \hat{\underline{\omega}}^N_{IN} - \underline{\omega}^N_{IN} \qquad \delta\underline{g}^N_P \equiv \hat{\underline{g}}^N_P - \underline{g}^N_P \tag{19}$$

and the angular error $\underline{\gamma}^N$ in $\hat{C}^N_B$ is implicitly defined from [5 - Eq. (83)] as

$$\delta C^N_B \equiv \hat{C}^N_B - C^N_B = - \left[\left(\underline{\gamma}^N \times\right) + \frac{1}{2}\left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right) + \cdots\right] \hat{C}^N_B \tag{20}$$

where

B = Coordinate (body) frame aligned with nominal INS inertial sensor input axes.

E = Coordinate (earth) frame aligned with axes fixed to the earth.

I = Inertially non-rotating reference coordinate frame.

$\underline{a}_{SF}$ = Specific force (non-gravitational) acceleration measured by INS accelerometers.

$\underline{g}^N_P$ = Plumb-bob gravity that equals the sum of earth's gravitational mass attraction plus earth's rotation centripetal acceleration effect. Defined as such because $\underline{g}_P^N$ lies along the direction of a plumb-bob under zero velocity conditions).

The attitude misalignment angle $\underline{\gamma}^N$ in (18) is obtained by integrating the attitude error rate equation, which from [5 - Eq. (88)], is given to second order accuracy by

$$\dot{\underline{\gamma}}^N \approx - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}^N + \delta\underline{\omega}_{IN}^N + \frac{1}{2}\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B + \delta\underline{\omega}_{IN}^N\right) \times \underline{\gamma}^N \qquad (21)$$

with

$$\delta\underline{\omega}_{IB}^N \equiv \hat{\underline{\omega}}_{IB}^N - \underline{\omega}_{IB}^N \qquad (22)$$

The linearized versions of (18) and (21) would be used by the Kalman filter in finding $\delta\underline{v}_{Lin}^N$ for the (6) and (17) measurement equations:

$$\begin{aligned} \delta\dot{\underline{v}}_{Lin}^N \approx\ & \hat{C}_B^N \, \delta\underline{a}_{SF}^B + \hat{\underline{a}}_{SF}^N \times \underline{\gamma}_{Lin}^N + \delta\underline{g}_P^N \\ & - \left(\delta\underline{\omega}_{IN}^N + \delta\underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \delta\underline{v}_{Lin}^N \end{aligned} \qquad (23)$$

$$\dot{\underline{\gamma}}_{Lin}^N = - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}_{Lin}^N + \delta\underline{\omega}_{IN}^N \qquad (24)$$

## INTEGRAL FORM OF THE ATTITUDE AND VELOCITY ERROR RATE EQUATIONS

In addition to differences between the form of the observation and measurement equation discussed previously, the modified velocity matching approach uses a different definition for attitude error $\underline{\gamma}^N$ than the traditional approach. For the modified approach, the initial value of $\underline{\gamma}^N$ is defined to have a zero heading error component. Thus, at Fine Alignment initialization, the INS initial attitude data for the modified approach is defined to only include horizontal components. The $\beta$ parameters in Eqs. (16) and (17) are the devices used to account for INS initial heading misalignment from a definable nominal reference frame (e.g., the N* frame used to provide velocity reference data to the alignment process). In contrast, the traditional alignment approach defines initial $\underline{\gamma}^N$ as containing both horizontal and vertical components. Mathematically then, the integral of (21) for the Observation Equation is given in general for the two approaches by:

$$\underline{\gamma}_{Trd}^N = \underline{\gamma}_{0_H}^N + \gamma_{0_Z}^N \, \underline{u}_Z^N + \int_0^t \dot{\underline{\gamma}}^N \, dt \qquad (25)$$

$$\underline{\gamma}^{N}_{Mod} = \underline{\gamma}^{N}_{0_H} + \int_0^t \dot{\underline{\gamma}}^{N} \, dt \qquad (26)$$

where

$0$ = Subscript indicating value of associated parameter at start of Fine Alignment.
$\gamma^{N}_{0_Z}$ = N frame Z axis component of $\underline{\gamma}^{N}_{0}$.

and $\dot{\underline{\gamma}}^{N}$ is given by (21). The equivalent linearized forms of (25) and (26) for Kalman filter design are

$$\underline{\gamma}^{N}_{Trd_{Lin}} = \underline{\gamma}^{N}_{0_H} + \gamma^{N}_{0_Z} \, \underline{u}^{N}_{Z} + \int_0^t \dot{\underline{\gamma}}^{N}_{Lin} \, dt \qquad (27)$$

$$\underline{\gamma}^{N}_{Mod_{Lin}} = \underline{\gamma}^{N}_{0_H} + \int_0^t \dot{\underline{\gamma}}^{N}_{Lin} \, dt \qquad (28)$$

with (24) for $\dot{\underline{\gamma}}^{N}_{Lin}$.

Velocity initialization at the start of Fine Alignment will be the same for the traditional and modified alignment approaches, i.e., both using input reference velocity. The traditional approach delivers reference velocity in N frame coordinates to the INS so that

$$\delta\underline{v}^{N}_{0_{Trd}} = \hat{\underline{v}}^{N}_{0_{Trd}} - \underline{v}^{N}_{0_{Trd}} = \hat{\underline{v}}^{N}_{Ref_0} - \left(\hat{\underline{v}}^{N}_{Ref_0} - \delta\underline{v}^{N}_{Ref_0}\right) = \delta\underline{v}^{N}_{Ref_0} \qquad (29)$$

With (29), the integral of (18) for the traditional approach then becomes

$$\delta\underline{v}^{N}_{Trd} = \delta\underline{v}^{N}_{0_{Trd}} + \int_0^t \delta\dot{\underline{v}}^{N}_{Trd} \, dt = \delta\underline{v}^{N}_{Ref_0} + \int_0^t \delta\dot{\underline{v}}^{N}_{Trd} \, dt \qquad (30)$$

Velocity error rate $\delta\dot{\underline{v}}^{N}_{Trd}$ in (30) is provided by (18), but with $\underline{\gamma}^{N}$ replaced by $\underline{\gamma}^{N}_{Trd}$ from (25).

For the modified approach, reference velocity is provided to the INS in N* coordinates. Thus, its use for $\hat{\underline{v}}^{N}$ initialization will contain an additional error due to the initially unknown heading variation between N and N* (i.e., $\hat{C}^{N}_{N*_0} = I$):

$$
\begin{aligned}
\underline{v}^{N}_{0_{Mod}} &= \hat{\underline{v}}^{N}_{0_{Mod}} - \delta\underline{v}^{N}_{0_{Mod}} = \hat{\underline{v}}^{N*}_{Ref_0} - \delta\underline{v}^{N}_{0_{Mod}} \\
&= C^{N}_{N*_0}\, \underline{v}^{N*}_{Ref_0} = \left(\hat{C}^{N}_{N*_0} - \delta C^{N}_{N*_0}\right)\left(\hat{\underline{v}}^{N*}_{Ref_0} - \delta\underline{v}^{N*}_{Ref_0}\right) \\
&\approx \hat{C}^{N}_{N*_0}\, \hat{\underline{v}}^{N*}_{Ref_0} - \hat{C}^{N}_{N*_0}\, \delta\underline{v}^{N*}_{Ref_0} - \delta C^{N}_{N*_0}\, \hat{\underline{v}}^{N*}_{Ref_0} \\
&= \hat{\underline{v}}^{N*}_{Ref_0} - \delta\underline{v}^{N*}_{Ref_0} - \delta C^{N}_{N*_0}\, \hat{\underline{v}}^{N*}_{Ref_0}
\end{aligned}
\tag{31}
$$

Thus,

$$
\delta\underline{v}^{N}_{0_{Mod}} = \delta\underline{v}^{N*}_{Ref_0} + \delta C^{N}_{N*_0}\, \hat{\underline{v}}^{N*}_{Ref_0}
\tag{32}
$$

Using (13) for $\delta C^{N}_{N*_0}$ in (32) while recognizing that $\beta$ is unknown at Fine Alignment initiation (i.e., $\hat{\beta}_0 = 0$), gives

$$
\begin{aligned}
\delta\underline{v}^{N}_{0_{Mod}} &= \delta\underline{v}^{N*}_{Ref_0} + \left[\delta\beta_0 \left(\underline{u}^{N}_{Z} \times\right) - \frac{1}{2}\, \delta\beta_0^2 \left(\underline{u}^{N}_{Z} \times\right)\left(\underline{u}^{N}_{Z} \times\right) + \cdots \right] \hat{\underline{v}}^{N*}_{Ref_0} \\
&= \delta\underline{v}^{N*}_{Ref_0} - \delta\beta_0\, \hat{\underline{v}}^{N*}_{Ref_{H/0}} \times \underline{u}^{N}_{Z} + \frac{1}{2}\, \delta\beta_0^2\, \hat{\underline{v}}^{N*}_{Ref_{H/0}} + \cdots
\end{aligned}
\tag{33}
$$

With (33), the integral of (18) for the modified approach then becomes

$$
\begin{aligned}
\delta\underline{v}^{N}_{Mod} &= \delta\underline{v}^{N}_{0_{Mod}} + \int_0^t \delta\dot{\underline{v}}^{N}_{Mod}\, dt \\
&= \delta\underline{v}^{N*}_{Ref_0} - \delta\beta_0\, \hat{\underline{v}}^{N*}_{Ref_{H/0}} \times \underline{u}^{N}_{Z} + \frac{1}{2}\, \delta\beta_0^2\, \hat{\underline{v}}^{N*}_{Ref_{H/0}} + \int_0^t \delta\dot{\underline{v}}^{N}_{Mod}\, dt + \cdots
\end{aligned}
\tag{34}
$$

Velocity error rate $\delta\dot{\underline{v}}^{N}_{Mod}$ in (34) is provided by (18), but with $\underline{\gamma}^{N}$ replaced by $\underline{\gamma}^{N}_{Mod}$ from (26).

## COMPARISON BETWEEN THE TRADITIONAL AND MODIFIED KALMAN ALIGNMENT APPROACHES

Eqs. (25), (26), (30), and (34) substituted in (5), (6), (16), and (17) provide a set of traditional and modified observation/measurement equations for second order impact performance comparisons.

### Traditional Measurement Equation

Consider the (6) traditional Measurement Equation. During the early phases of Fine Alignment, error rate buildup has a short time to develop, thus the errors are dominated by their initial values. With (25) for the traditional approach, attitude error rate (24) approximates as

$$\dot{\underline{\gamma}}^N_{Trd_{Lin}} \approx - \hat{\underline{\omega}}^N_{IN_0} \times \left( \underline{\gamma}^N_{0_H} + \gamma^N_{0_Z} \, \underline{u}^N_Z \right) + \cdots = - \hat{\underline{\omega}}^N_{IN_0} \times \underline{\gamma}^N_{0_H} - \gamma^N_{0_Z} \, \hat{\underline{\omega}}^N_{IN_0} \times \underline{u}^N_Z + \cdots \qquad (35)$$

with which (27) becomes

$$\underline{\gamma}^N_{Trd_{Lin}} = \underline{\gamma}^N_{0_H} + \gamma^N_{0_Z} \, \underline{u}^N_Z - \left( \hat{\underline{\omega}}^N_{IN_0} \, t \right) \times \underline{\gamma}^N_{0_H} - \gamma^N_{0_Z} \left( \hat{\underline{\omega}}^N_{IN_{H/0}} \, t \right) \times \underline{u}^N_Z + \cdots \qquad (36)$$

The velocity error for traditional Measurement Equation (6) is then obtained by substituting (36) in (23) and integrating. First, (23) becomes

$$\delta \dot{\underline{v}}^N_{Trd_{Lin}} = \hat{\underline{a}}^N_{SF} \times \left[ \underline{\gamma}^N_{0_H} + \gamma^N_{0_Z} \, \underline{u}^N_Z - \left( \hat{\underline{\omega}}^N_{IN_0} \, t \right) \times \underline{\gamma}^N_{0_H} - \gamma^N_{0_Z} \left( \hat{\underline{\omega}}^N_{IN_{H/0}} \, t \right) \times \underline{u}^N_Z \right] + \cdots \qquad (37)$$

Expanding finds for the horizontal components:

$$\begin{aligned} \delta \dot{\underline{v}}^N_{Trd_{Lin/H}} &= \left\{ \left( \hat{\underline{a}}^N_{SF_H} + \hat{a}^N_{SF_Z} \, \underline{u}^N_Z \right) \times \left[ \begin{array}{c} \underline{\gamma}^N_{0_H} + \gamma^N_{0_Z} \, \underline{u}^N_Z - \left( \hat{\underline{\omega}}^N_{IN_0} \, t \right) \times \underline{\gamma}^N_{0_H} \\ - \gamma^N_{0_Z} \left( \hat{\underline{\omega}}^N_{IN_{H/0}} \, t \right) \times \underline{u}^N_Z \end{array} \right] \right\}_H + \cdots \\ &= \gamma^N_{0_Z} \, \hat{\underline{a}}^N_{SF_H} \times \underline{u}^N_Z + \hat{a}^N_{SF_Z} \times \underline{\gamma}^N_{0_H} - \gamma^N_{0_Z} \, \hat{a}^N_{SF_Z} \, \underline{u}^N_Z \times \left[ \left( \hat{\underline{\omega}}^N_{IN_{H/0}} \, t \right) \times \underline{u}^N_Z \right] + \cdots \\ &= \gamma^N_{0_Z} \, \hat{\underline{a}}^N_{SF_H} \times \underline{u}^N_Z + \hat{a}^N_{SF_Z} \times \underline{\gamma}^N_{0_H} - \gamma^N_{0_Z} \left( \hat{a}^N_{SF_Z} \, t \right) \hat{\underline{\omega}}^N_{IN_{H/0}} + \cdots \end{aligned} \qquad (38)$$

Integrating (38) with (29) for initialization obtains

$$\begin{aligned} \delta \underline{v}^N_{Trd_{Lin/H}} &= \delta \underline{v}^N_{Ref_{H/0}} + \left( \int_0^t \hat{a}^N_{SF_Z} \, dt \right) \times \underline{\gamma}^N_{0_H} \\ &+ \gamma^N_{0_Z} \left( \int_0^t \hat{\underline{a}}^N_{SF_H} \, dt \right) \times \underline{u}^N_Z - \gamma^N_{0_Z} \left( \int_0^t \hat{a}^N_{SF_Z} \, t \, dt \right) \hat{\underline{\omega}}^N_{IN_{H/0}} + \cdots \\ &= \delta \underline{v}^N_{Ref_{H/0}} + \left( \int_0^t \hat{a}^N_{SF_Z} \, dt \right) \times \underline{\gamma}^N_{0_H} \\ &+ \gamma^N_{0_Z} \left( \hat{\underline{v}}^N_H - \hat{\underline{v}}^N_{H_0} \right) \times \underline{u}^N_Z - \gamma^N_{0_Z} \left( \int_0^t \hat{a}^N_{SF_Z} \, t \, dt \right) \hat{\underline{\omega}}^N_{IN_{H/0}} + \cdots \end{aligned} \qquad (39)$$

Substituting (39) into (6) then finds for the horizontal components of the traditional Measurement Equation:

$$\begin{aligned} \underline{z}^N_{Trd_H} &= \left( \int_0^t \hat{a}^N_{SF_Z} \, dt \right) \times \underline{\gamma}^N_{0_H} - \left( \delta \underline{v}^N_{Ref_H} - \delta \underline{v}^N_{Ref_{H/0}} \right) \\ &+ \gamma^N_{0_Z} \left( \hat{\underline{v}}^N_H - \hat{\underline{v}}^N_{H_0} \right) \times \underline{u}^N_Z - \gamma^N_{0_Z} \left( \int_0^t \hat{a}^N_{SF_Z} \, t \, dt \right) \hat{\underline{\omega}}^N_{IN_{H/0}} + \cdots \end{aligned} \qquad (40)$$

Modified Measurement Equation

Following the previous methodology but using (26) for the modified approach, (24) approximates as

$$\dot{\underline{\gamma}}^N_{Mod_{Lin}} \approx - \hat{\underline{\omega}}^N_{IN_0} \times \underline{\gamma}^N_{0_H} + \delta\underline{\omega}^N_{IN_0} \; \cdots \quad (41)$$

Note that in contrast with (35) for the traditional approach, (41) for the modified approach has retained $\delta\underline{\omega}^N_{IN_0}$ as of significance during early Fine Alignment. This is because $\hat{\underline{\omega}}^N_{IN_0}$ is derived from N* velocity and position reference data, provided to the INS and initially treated as being along N frame coordinates (i.e., not accounting for the initially unknown $\beta$ misalignment between N and N*). Thus, $\beta_0 = \beta = \hat{\beta}_0 - \delta\beta_0 = - \delta\beta_0$, and from (12):

$$\begin{aligned} C^N_{N^*_0} &= I + \beta \left(\underline{u}^N_Z \times\right) + \cdots = I - \delta\beta_0 \left(\underline{u}^N_Z \times\right) + \cdots \\ \delta\underline{\omega}^N_{IN_0} &\equiv \hat{\underline{\omega}}^N_{IN_0} - \underline{\omega}^N_{IN_0} = \underline{\omega}^{N^*}_{IN_{Ref/0}} - C^N_{N^*_0} \, \underline{\omega}^{N^*}_{IN_{Ref/0}} \\ &= \left(I - C^N_{N^*_0}\right) \underline{\omega}^{N^*}_{IN_{Ref/0}} = - \delta\beta_0 \, \underline{\omega}^{N^*}_{IN_{Ref/0}} \times \underline{u}^N_Z + \cdots \end{aligned} \quad (42)$$

With $\delta\underline{\omega}^N_{IN_0}$ from (42), Eq. (41) then becomes

$$\dot{\underline{\gamma}}^N_{Mod_{Lin}} = - \hat{\underline{\omega}}^N_{IN_0} \times \underline{\gamma}^N_{0_H} - \delta\beta_0 \, \hat{\underline{\omega}}^N_{IN_{H/0}} \times \underline{u}^N_Z + \cdots \quad (43)$$

Substituting (43) into (26) yields for the modified approach:

$$\underline{\gamma}^N_{Mod_{Lin}} = \underline{\gamma}^N_{0_H} - \left(\hat{\underline{\omega}}^N_{IN_0} \, t\right) \times \underline{\gamma}^N_{0_H} - \delta\beta_0 \left(\hat{\underline{\omega}}^N_{IN_{H/0}} \, t\right) \times \underline{u}^N_Z + \cdots \quad (44)$$

Comparing (44) with (36) finds that the results are similar except for the addition of $\gamma^N_{0_Z} \, \underline{u}^N_Z$ in traditional Eq. (36), and for the heading misalignment error $\delta\beta_0$ replacing the equivalent $\gamma^N_{0_Z}$ in the product with $\left(\hat{\underline{\omega}}^N_{IN_{H/0}} \, t\right)$. As will be apparent subsequently, elimination of $\gamma^N_{0_Z} \, \underline{u}^N_Z$ with the modified approach removes the second order error problem exhibited by the traditional approach under benign flight conditions [5].

The velocity error for modified Measurement Equation (17) is then obtained by substituting (44) in (23) and integrating. First, (23) becomes

$$\delta \dot{\underline{v}}^N_{Mod_{Lin}} = \hat{\underline{a}}^N_{SF} \times \left[ \underline{\gamma}^N_{0_H} - \left( \hat{\underline{\omega}}^N_{IN_0}\, t \right) \times \underline{\gamma}^N_{0_H} - \delta\beta_0 \left( \hat{\underline{\omega}}^N_{IN_{H/0}}\, t \right) \times \underline{u}^N_Z \right] + \cdots \tag{45}$$

Expanding finds for the horizontal components:

$$\begin{aligned} \delta \dot{\underline{v}}^N_{Mod_{Lin/H}} &= \left\{ \left( \hat{\underline{a}}^N_{SF_H} + \hat{a}^N_{SF_Z}\, \underline{u}^N_Z \right) \times \begin{pmatrix} \underline{\gamma}^N_{0_H} - \left( \hat{\underline{\omega}}^N_{IN_0}\, t \right) \times \underline{\gamma}^N_{0_H} \\ - \delta\beta_0 \left( \hat{\underline{\omega}}^N_{IN_{H/0}}\, t \right) \times \underline{u}^N_Z \end{pmatrix} \right\}_H + \cdots \\ &= \hat{a}^N_{SF_Z} \times \underline{\gamma}^N_{0_H} - \delta\beta_0\, \hat{a}^N_{SF_Z}\, \underline{u}^N_Z \times \left[ \left( \hat{\underline{\omega}}^N_{IN_{H/0}}\, t \right) \times \underline{u}^N_Z \right] + \cdots \\ &= \hat{a}^N_{SF_Z} \times \underline{\gamma}^N_{0_H} - \delta\beta_0 \left( \hat{a}^N_{SF_Z}\, t \right) \hat{\underline{\omega}}^N_{IN_{H/0}} + \cdots \end{aligned} \tag{46}$$

Integrating (46) using the linearized form of (33) for initialization obtains

$$\begin{aligned} \delta \underline{v}^N_{Mod_{Lin/H}} &= \delta \underline{v}^{N*}_{Ref_0} - \delta\beta_0\, \hat{\underline{v}}^{N*}_{Ref_{H/0}} \times \underline{u}^N_Z \\ &+ \left( \int_0^t \hat{a}^N_{SF_Z}\, dt \right) \times \underline{\gamma}^N_{0_H} - \delta\beta_0 \left( \int_0^t \hat{a}^N_{SF_Z}\, t\, dt \right) \hat{\underline{\omega}}^N_{IN_{H/0}} + \cdots \end{aligned} \tag{47}$$

Then, substituting (47) into (17) finds for the horizontal components of the modified measurement equation:

$$\begin{aligned} \underline{z}^N_{Mod_H} &= \left( \int_0^t \hat{a}^N_{SF_Z}\, dt \right) \times \underline{\gamma}^N_{0_H} - \left( \delta \underline{v}^{N*}_{Ref} - \delta \underline{v}^{N*}_{Ref_0} \right) \\ &+ \delta\beta_0 \left( \hat{\underline{v}}^{N*}_{Ref_H} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right) \times \underline{u}^N_Z - \delta\beta_0 \left( \int_0^t \hat{a}^N_{SF_Z}\, t\, dt \right) \hat{\underline{\omega}}^N_{IN_{H/0}} + \cdots \end{aligned} \tag{48}$$

Comparing (48) for the modified measurement with (40) for the traditional measurement shows that the two are equivalent with $\delta\beta_0$ and $\gamma^N_{Z_0}$ both representing initial INS heading error. The difference between the two approaches becomes apparent when the second order errors in the measurements are compared (discussed subsequently).

## Second Order Errors In the Measurements

The second order errors introduced by the linearization process that led to traditional Measurement Equation (40), can be determined by repeating the process leading to (40), but without linearization; then comparing with the (40) linearized form. The equivalent non-linearized version for comparison with (40) would then be the (5) form of the traditional observation, with a non-linearized version of $\delta \underline{v}^N_{Trd}$ substituted for $\delta \underline{v}^N$. First note, as in [5], that the $\left( \hat{\underline{a}}^N_{SF} \times \underline{\gamma}^N \right) \times \underline{\gamma}^N$ in (18) expands as

$$\left(\underline{\hat{a}}_{SF}^{N} \times \underline{\gamma}^{N}\right) \times \underline{\gamma}^{N} = \underline{\gamma}^{N} \left(\underline{\hat{a}}_{SF}^{N} \cdot \underline{\gamma}^{N}\right) - \underline{\hat{a}}_{SF}^{N} \left(\gamma^{2}\right)$$
$$= \left(\underline{\gamma}_{H}^{N} + \gamma_{Z}^{N}\, \underline{u}_{Z}^{N}\right) \left(\underline{\hat{a}}_{SF_{H}}^{N} \cdot \underline{\gamma}_{H}^{N} + \hat{a}_{SF_{Z}}^{N}\, \gamma_{Z}^{N}\right) + \cdots = \hat{a}_{SF_{Z}}^{N}\, \gamma_{Z}^{N}\, \underline{\gamma}_{H}^{N} + \cdots \tag{49}$$

so that integrating (18) with initial condition (29) obtains for $\delta\underline{v}_{Trd}^{N}$ during the early portion of Fine Alignment:

$$\delta\underline{v}_{Trd}^{N} = \delta\underline{v}_{Trd_{Lin}}^{N} - \gamma_{Z_0}^{N} \frac{1}{2} \left(\int_0^t \hat{a}_{SF_Z}^{N}\, dt\right) \underline{\gamma}_{H}^{N} + \cdots \tag{50}$$

With (50), the non-linearized observation for comparison with (40) becomes

$$\underline{\widehat{M}}_{Trd_H}^{N} = \underline{z}_{Trd_H}^{N} - \gamma_{Z_0}^{N} \frac{1}{2} \left(\int_0^t \hat{a}_{SF_Z}^{N}\, dt\right) \underline{\gamma}_{H_0}^{N} + \cdots$$
$$= \left(\int_0^t \hat{a}_{SF_Z}^{N}\, dt\right) \times \underline{\gamma}_{0_H}^{N} - \left(\delta\underline{v}_{Ref_H}^{N} - \delta\underline{v}_{Ref_{H/0}}^{N}\right) \tag{51}$$
$$+ \gamma_{0_Z}^{N} \left[\left(\underline{\hat{v}}_{H}^{N} - \underline{\hat{v}}_{H_0}^{N}\right) - \frac{1}{2} \left(\int_0^t \hat{a}_{SF_Z}^{N}\, dt\right) \underline{\gamma}_{H_0}^{N}\right] \times \underline{u}_{Z}^{N} - \gamma_{0_Z}^{N} \left(\int_0^t \hat{a}_{SF_Z}^{N}\, t\, dt\right) \underline{\hat{\omega}}_{IN_{H/0}}^{N} + \cdots$$

In (51), only the primary second order term contributing to performance anomalies [5] is shown.

As in [5], Eq. (51) shows that the $\frac{1}{2} \left(\int_0^t \hat{a}_{SF_Z}^{N}\, dt\right) \underline{\gamma}_{H_0}^{N}$ coefficient in second order term $\gamma_{0_Z}^{N} \frac{1}{2} \left(\int_0^t \hat{a}_{SF_Z}^{N}\, dt\right) \underline{\gamma}_{H_0}^{N}$ appears as an addition to the $\left(\underline{\hat{v}}_{H}^{N} - \underline{\hat{v}}_{H_0}^{N}\right)$ maneuver coefficient, both multiplying heading error $\gamma_{0_Z}^{N}$. Under normal maneuvering, the second order term is negligible. However, under benign flight conditions (i.e., when $\left(\underline{\hat{v}}_{H}^{N} - \underline{\hat{v}}_{H_0}^{N}\right)$ is small), the second order term becomes significant, being interpreted by the Kalman filter as true maneuvering in the Kalman linearized $\underline{z}_{Trd_H}^{N}$ measurement equation. The result would be a mis-estimation of heading error $\gamma_{0_Z}^{N}$. Reference [5] mitigated the effect by modeling the $\gamma_{0_Z}^{N} \frac{1}{2} \left(\int_0^t \hat{a}_{SF_Z}^{N}\, dt\right) \underline{\gamma}_{H_0}^{N}$ second order term as adaptive artificial noise in the $\underline{z}_{Trd_H}^{N}$ measurement.

For the modified observation/measurement equation approach, because $\gamma_{0_Z}^{N}$ in (49) is defined to be zero (being replaced by $\delta\beta_0$), the equivalent to the (51) derivation yields

$$\underline{\widehat{M}}^N_{Mod_H} = \underline{z}^N_{Mod_H} + \cdots$$

$$= \left(\int_0^t \hat{\underline{a}}^N_{SF_Z}\, dt\right) \times \underline{\gamma}^N_{0_H} - \left(\delta\underline{v}^{N*}_{Ref} - \delta\underline{v}^{N*}_{Ref_0}\right) \tag{52}$$

$$+ \delta\beta_0 \left(\hat{\underline{v}}^{N*}_{Ref_H} - \hat{\underline{v}}^{N*}_{Ref_{H/0}}\right) \times \underline{u}^N_Z - \delta\beta_0 \left(\int_0^t \hat{\underline{a}}^N_{SF_Z}\, t\, dt\right) \hat{\underline{\omega}}^N_{IN_{H/0}} + \cdots$$

Thus, for the modified approach, the $\underline{\widehat{M}}^N_{Mod_H}$ observation input to the Kalman filter does not contain the troublesome $\gamma^N_{0_Z} \frac{1}{2} \left(\int_0^t \hat{a}^N_{SF_Z}\, dt\right) \underline{\gamma}^N_{H_0}$ second order term in (51), and measurement $\underline{z}^N_{Mod_H}$ (the linearized form of $\underline{\widehat{M}}^N_{Mod_H}$ used in calculating the Kalman gains), accurately models the error in $\underline{\widehat{M}}^N_{Mod_H}$ to second order accuracy.

## SUMMARY

The equations for implementing the modified velocity matching measurement approach are (7), (12), (15), (17), (28) and (33) summarized and renumbered next.

Observation Equation

$$\underline{\widehat{M}}^N_{Mod} = \hat{\underline{v}}^N - \widehat{C}^N_{N*}\, \hat{\underline{v}}^{N*}_{Ref} \tag{53}$$

$$\widehat{C}^N_{N*} = I + \hat{\beta}\left(\underline{u}^N_Z \times\right) + \frac{1}{2}\hat{\beta}^2 \left(\underline{u}^N_Z \times\right)\left(\underline{u}^N_Z \times\right) + \cdots \tag{54}$$

Kalman Filter Measurement Equation

$$\underline{z}^N_{Mod} = \delta\underline{v}^N_{Lin} - \delta\underline{v}^{N*}_{Ref} + \delta\beta\, \hat{\underline{v}}^{N*}_{Ref_H} \times \underline{u}^N_Z \tag{55}$$

Addition To Kalman Filter Error State Dynamic Equations

$$\delta\dot{\beta} = 0 \tag{56}$$

Addition To INS Navigation Equations

$$\dot{\hat{\beta}} = 0 \tag{57}$$

Conditions For Initializing Kalman Filter Covariance Matrix

$$\delta\underline{v}^N_{0_{Lin/Mod}} = \delta\underline{v}^{N*}_{Ref_0} - \delta\beta_0\, \hat{\underline{v}}^{N*}_{Ref_{H/0}} \times \underline{u}^N_Z \tag{58}$$

$$\underline{\gamma}^N_{0_{Mod/Lin}} = \underline{\gamma}^N_{0_H} \tag{59}$$

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