# INERTIAL SENSOR STABILITY EVALUATION BY DUAL SENSOR TESTING

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ABSTRACT

This article introduces the concept of dual sensor testing as a simple method for evaluating the stability of inertial sensors. The concept mounts two sensors of the same design configuration on a common mount with their respective axes parallel. Test measurements are the difference between sensor outputs under different orientations and environmental exposures. Variations in like-measurements over time determines the stability of the sensor error sources. A key advantage of the technique is the elimination of the need for sophisticated and expensive test equipment to generate precise measures of sensor stability. As a specific example, the article analytically applies the technique to a generic dual accelerometer configuration showing how test equipment inaccuracy generates only second order errors in the stability measurements. The article also demonstrates how stability measurement accuracy can be enhanced by using the first set of measurements to calibrate successive measurements.

## INTRODUCTION

The principle factor in determining the applicability of a new inertial sensor is the stability of its performance parameters over the environment of its intended use. This is the only error source remaining following an accurate calibration. This article describes a simple method of measuring inertial sensor stability without the use of elaborate and expensive test fixturing. The method implements the basic concept of dual sensor testing in which two samples of the instrument to be tested are affixed to a common mount with instrument axes parallel. The stability test measurement is the difference between sensor outputs. The measurement will be zero under zero sensor error conditions, hence, in the presence of sensor errors, will measure the difference between the sensor error parameters. As a result, test fixturing errors induced into the measurement will be second order because the dominant sensor input will be eliminated in the dual sensor output subtraction process. Thus, test instrumentation accuracy requirements are significantly reduced making it possible to achieve high accuracy multi-position stability measurements (e.g., for accelerometers) with the sensors affixed within a simple mounting cube. Exposing the cube to environments at different orientations enables accurate sensor error stability evaluation at these orientations.

This article analyzes a dual test setup for a generic accelerometer model, analytically deriving the difference in accelerometer output measurements as a function of analytically modeled sensor errors for the six possible cube orientations to vertical (bias, scale factor error, scale factor asymmetry, linear scale factor asymmetry, misalignment to the mounting cube, and mounting cube misalignment to vertical induced by test instrumentation). Results demonstrate that each measurement evaluates the key error source differences to first order plus second order terms dependent on test fixture misalignment effects. The article shows how the six measurements can be analytically combined to evaluate the difference in each sensor error characteristic, and to compensate these results using calibration values determined at the start of testing. The calibration values comprise the first set of sensor error characteristics determined from the dual sensor difference measurements at test series start.

## TEST CUBE DEFINITION

Consider a metal cube for testing the stability of accelerometer error parameters over expected environments. Define a cube coordinate frame *D* having x, y, z right-handed orthogonal axes parallel to the cube edges. Further, define six test orientations of the cube, 1 with x axis up, 2 with x axis down, and four with x axis horizontal. For the four x axis horizontal orientations, orientation 3 has z axis down, 4 has y axis down, 5 has z axis up, and 6 has y axis up. Define nominal conditions as having one of the cube faces perpendicular to plumb-bob gravity. Based on the above definitions, the following would be the specific force vector components (reaction to local gravity of magnitude *g*) in the *D* frame:

$$\underline{a}_1 = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} \quad \underline{a}_2 = \begin{bmatrix} -g \\ 0 \\ 0 \end{bmatrix} \quad \underline{a}_3 = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \quad \underline{a}_4 = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} \quad \underline{a}_5 = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \quad \underline{a}_6 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} \qquad (1)$$

## ACCELEROMETER MOUNTING WITHIN THE TEST CUBE

Now consider two "identical" accelerometers *a* and *b* mounted within the cube, each with the sensing axis (x) parallel to the cube x axis, and the cross axes (y and z) parallel to the cube y and z axes. Define the analytical model for the accelerometer *a* output as

$$\begin{gathered} a_{Out_{ai}} = a_{Bias_a} + \left(1 + L_{Sfer_a} + L_{Asym_a}\, a_{In_i} + L_{LinAsym_a}\, sign(a_{In_i})\right) a_{In_i} \\ a_{In_i} = \underline{u}_{In}^T\, C_{\theta_a}\, C_{\phi_i}\, \underline{a}_i \qquad \underline{u}_{In}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{gathered} \qquad (2)$$

where

$a_{Out_{ai}}$ = Output for accelerometer *a* at test orientation *i*.
$a_{In_i}$ = Nominal (true) specific force acceleration along accelerometer sensing axis at test orientation *i*.
$a_{Bias_a}$ = Accelerometer *a* bias error.

$L_{Sfer_a}$ = Accelerometer *a* scale factor error. May include a polynomial non-linearity function of $a_{In_i}^2$.

$L_{Asym_a}$ = Accelerometer *a* scale factor asymmetry. May include a polynomial non-linearity function of $a_{In_i}^2$.

$L_{LinAsym_a}$ = Accelerometer *a* linear scale factor asymmetry. Represents a shift in scale factor for positive compared to negative $a_{In_i}$ values.

$sign(a_{In_i})$ = Unity magnitude with sign (plus or minus) of the bracketed term.

$\underline{u}_{In}$ = Unit vector along the accelerometer sensing input axis.

$C_{\theta_a}$ = Direction cosine matrix representing small misalignments of the accelerometer *a* sensing axis from its nominal orientation parallel to test cube axis *x*.

$C_{\phi_i}$ = Direction cosine matrix representing small misalignments of the test cube from vertical at test orientation *i*.

Ref. [1, Sect. 3.2.2.1] shows that $C_{\theta_a}$ and $C_{\phi_i}$ in (2) can be expressed in terms of equivalent small angle rotation vectors as

$$C_{\theta_a} = I + \left(\underline{\theta}_a \times\right) + \frac{1}{2}\left(\underline{\theta}_a \times\right)^2 + \cdots \qquad \underline{\theta}_a = \begin{bmatrix} \theta_{ax} \\ \theta_{ay} \\ \theta_{az} \end{bmatrix} \qquad \left(\underline{\theta}_a \times\right) = \begin{bmatrix} 0 & -\theta_{az} & \theta_{ay} \\ \theta_{az} & 0 & -\theta_{ax} \\ -\theta_{ay} & \theta_{ax} & 0 \end{bmatrix}$$

$$\frac{1}{2}\left(\underline{\theta}_a \times\right)^2 = \frac{1}{2}\begin{bmatrix} -\left(\theta_{az}^2 + \theta_{ay}^2\right) & \theta_{ax}\,\theta_{ay} & \theta_{az}\,\theta_{ax} \\ \theta_{ax}\,\theta_{ay} & -\left(\theta_{ax}^2 + \theta_{az}^2\right) & \theta_{ay}\,\theta_{az} \\ \theta_{az}\,\theta_{ax} & \theta_{ay}\,\theta_{az} & -\left(\theta_{ay}^2 + \theta_{ax}^2\right) \end{bmatrix} \tag{3}$$

$$C_{\phi_i} = I + \left(\underline{\phi}_i \times\right) + \frac{1}{2}\left(\underline{\phi}_i \times\right)^2 + \cdots \qquad \underline{\phi}_i = \begin{bmatrix} \phi_{ix} \\ \phi_{iy} \\ \phi_{iz} \end{bmatrix} \qquad \left(\underline{\phi}_i \times\right) = \begin{bmatrix} 0 & -\phi_{iz} & \phi_{iy} \\ \phi_{iz} & 0 & -\phi_{ix} \\ -\phi_{iy} & \phi_{ix} & 0 \end{bmatrix}$$

$$\frac{1}{2}\left(\underline{\phi}_i \times\right)^2 = \frac{1}{2}\begin{bmatrix} -\left(\phi_{iz}^2 + \phi_{iy}^2\right) & \phi_{ix}\,\phi_{iy} & \phi_{iz}\,\phi_{ix} \\ \phi_{ix}\,\phi_{iy} & -\left(\phi_{ix}^2 + \phi_{iz}^2\right) & \phi_{iy}\,\phi_{iz} \\ \phi_{iz}\,\phi_{ix} & \phi_{iy}\,\phi_{iz} & -\left(\phi_{iy}^2 + \phi_{ix}^2\right) \end{bmatrix} \tag{4}$$

where

$\underline{\theta}_a$ = Small angle rotation vector equivalent of direction cosine matrix $C_{\theta_a}$.
$\underline{\phi}_i$ = Small angle rotation vector equivalent of direction cosine matrix $C_{\phi_i}$.

The analytical model for accelerometer *b* is identical to (2) – (3) with the *a* designation replaced by *b*.

TEST MEASUREMENTS

Test measurement $Z_i$ at test orientation *i* is the difference between the accelerometer *a* and *b* outputs:

$$Z_i = a_{Out_{ai}} - a_{Out_{bi}} \tag{5}$$

Substituting (1) and (2) – (4) into (5) finds to second order accuracy from (A-13) of Appendix A:

$$\begin{aligned}
Z_1 &= \left(a_{Bias_a} - a_{Bias_b}\right) + \begin{bmatrix} -0.5\left[\left(\theta_{az}^2 - \theta_{bz}^2\right) + \left(\theta_{ay}^2 - \theta_{by}^2\right)\right] \\ -(\theta_{az} - \theta_{bz})\phi_{1z} - \left(\theta_{ay} - \theta_{by}\right)\phi_{1y} + \left(L_{Sfer_a} - L_{Sfer_b}\right) \\ +\left(L_{Asym_a} - L_{Asym_b}\right) g + \left(L_{LinAsym_a} - L_{LinAsym_b}\right) \end{bmatrix} g \\
Z_2 &= \left(a_{Bias_a} - a_{Bias_b}\right) - \begin{bmatrix} -0.5\left[\left(\theta_{az}^2 - \theta_{bz}^2\right) + \left(\theta_{ay}^2 - \theta_{by}^2\right)\right] \\ -(\theta_{az} - \theta_{bz})\phi_{2z} - \left(\theta_{ay} - \theta_{by}\right)\phi_{2y} + \left(L_{Sfer_a} - L_{Sfer_b}\right) \\ -\left(L_{Asym_a} - L_{Asym_b}\right) g - \left(L_{LinAsym_a} - L_{LinAsym_b}\right) \end{bmatrix} g \\
Z_3 &= \left(a_{Bias_a} - a_{Bias_b}\right) - \begin{bmatrix} \left(\theta_{ay} - \theta_{by}\right) + (\theta_{az} - \theta_{bz})\phi_{3x} + 0.5(\theta_{az}\,\theta_{ax} - \theta_{bz}\,\theta_{bx}) \\ +\left(L_{Sfer_a} - L_{Sfer_b}\right)\,\phi_{3y} - \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left|\,\phi_{3y}\right| \end{bmatrix} g \\
Z_4 &= \left(a_{Bias_a} - a_{Bias_b}\right) - \begin{bmatrix} -(\theta_{az} - \theta_{bz}) + \left(\theta_{ay} - \theta_{by}\right)\phi_{4x} + 0.5\left(\theta_{ax}\,\theta_{ay} - \theta_{bx}\,\theta_{by}\right) \\ -\left(L_{Sfer_a} - L_{Sfer_b}\right)\,\phi_{4z} - \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left|\phi_{4z}\right| \end{bmatrix} g \\
Z_5 &= \left(a_{Bias_a} - a_{Bias_b}\right) + \begin{bmatrix} \left(\theta_{ay} - \theta_{by}\right) + (\theta_{az} - \theta_{bz})\phi_{5x} + 0.5(\theta_{az}\,\theta_{ax} - \theta_{bz}\,\theta_{bx}) \\ +\left(L_{Sfer_a} - L_{Sfer_b}\right)\,\phi_{5y} + \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left|\,\phi_{5y}\right| \end{bmatrix} g \\
Z_6 &= \left(a_{Bias_a} - a_{Bias_b}\right) + \begin{bmatrix} -(\theta_{az} - \theta_{bz}) + \left(\theta_{ay} - \theta_{by}\right)\phi_{6x} + 0.5\left(\theta_{ax}\,\theta_{ay} - \theta_{bx}\,\theta_{by}\right) \\ -\left(L_{Sfer_a} - L_{Sfer_b}\right)\,\phi_{6z} + \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left|\phi_{6z}\right| \end{bmatrix} g
\end{aligned} \tag{6}$$

Eqs. (6) show that all accelerometer error inputs to the Z measurements appear as differences between the *a* and *b* accelerometer error characteristics. Eqs. (6) also show that the $\underline{\phi}_i$ misalignments of the test cube from vertical only impact the Z measurements to second order (as products with the accelerometer *a* and *b* error differences). Thus, for *a*, *b* accelerometer error characteristics that are equal (or zero), the Z measurements would be zero regardless of test cube misalignments to vertical (as was expected).

Combining appropriate (6) terms with rearrangement, (A-15) of Appendix A shows that to second order accuracy,

$$\begin{aligned}
\left(a_{Bias_a} - a_{Bias_b}\right)/g &= \left(Z_5 + Z_3\right)/2g - 0.5\left(\theta_{az} - \theta_{bz}\right)\left(\phi_{5x} - \phi_{3x}\right) \\
&- 0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)\left(\phi_{5y} - \phi_{3y}\right) - 0.5\left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left(\left|\phi_{5y}\right| + \left|\phi_{3y}\right|\right) \\
\left(L_{Sfer_a} - L_{Sfer_b}\right) &= \left(Z_1 - Z_2\right)/2g + 0.5\left(\theta_{az} - \theta_{bz}\right)\left(\phi_{1z} - \phi_{2z}\right) + 0.5\left(\theta_{ay} - \theta_{by}\right)\left(\phi_{1y} - \phi_{2y}\right) \\
\left(L_{Asym_a} - L_{Asym_b}\right) g + \left(L_{LinAsym_a} - L_{LinAsym_b}\right) &= \left(Z_1 + Z_2 - Z_5 - Z_3\right)/2g \\
&+ 0.5\left(\theta_{az} - \theta_{bz}\right)\left(\phi_{1z} - \phi_{2z} + \phi_{5x} - \phi_{3x}\right) + 0.5\left(\theta_{ay} - \theta_{by}\right)\left(\phi_{1y} - \phi_{2y}\right) \\
&+ 0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)\left(\phi_{5y} - \phi_{3y}\right) + 0.5\left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left(\left|\phi_{5y}\right| + \left|\phi_{3y}\right|\right) \\
\left(\theta_{ay} - \theta_{by}\right) &= \left(Z_5 - Z_3\right)/2g - 0.5\left(\theta_{az} - \theta_{bz}\right)\left(\phi_{5x} + \phi_{3x}\right) - 0.5\left(\theta_{az}\,\theta_{ax} - \theta_{bz}\,\theta_{bx}\right) \\
&- 0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)\left(\phi_{5y} + \phi_{3y}\right) \\
\left(\theta_{az} - \theta_{bz}\right) &= \left(Z_4 - Z_6\right)/2g + 0.5\left(\theta_{ay} - \theta_{by}\right)\left(\phi_{4x} + \phi_{6x}\right) + 0.5\left(\theta_{ax}\,\theta_{ay} - \theta_{bx}\,\theta_{by}\right) \\
&- 0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)\left(\phi_{4z} + \phi_{6x}\right)
\end{aligned} \tag{7}$$

Based on (7), the set of six (*i*) measurements represented in (1) (for each cube face up), provides the following for estimating sensor error differences:

$$\begin{aligned}
\left(\hat{a}_{Bias_a} - \hat{a}_{Bias_b}\right)/g &= \left(Z_5 + Z_3\right)/2g \\
\left(\hat{L}_{Sfer_a} - \hat{L}_{Sfer_b}\right) &= \left(Z_1 - Z_2\right)/2g \\
\left[\left(\hat{L}_{Asym_a} - \hat{L}_{Asym_b}\right) g + \left(\hat{L}_{LinAsym_a} - \hat{L}_{LinAsym_b}\right)\right] &= \left(Z_1 + Z_2 - Z_5 - Z_3\right)/2\mathrm{g} \\
\left(\hat{\theta}_{ay} - \hat{\theta}_{by}\right) = \left(Z_5 - Z_3\right)/2g \qquad & \left(\hat{\theta}_{az} - \hat{\theta}_{bz}\right) = \left(Z_4 - Z_6\right)/2g
\end{aligned} \tag{8}$$

where $\widehat{(\,)}$ is the estimated value for the parameter in brackets.

Prior to formal stability testing, initial test results can be calibrated by first making the $z_i$ measurements at room temperature on a leveled flat plate at room. These calibration measurements would then be used to correct the (8) estimates during the subsequent stability tests:

$$
\begin{gathered}
\delta\left(a_{Bias_a} - a_{Bias_b}\right) = \left(\hat{a}_{Bias_a} - \hat{a}_{Bias_b}\right) - \left(a_{Bias_a} - a_{Bias_b}\right)_{Cal} \\
\delta\left(L_{Sfer_a} - L_{Sfer_b}\right) = \left(\hat{L}_{Sfer_a} - \hat{L}_{Sfer_b}\right) - \left(L_{Sfer_a} - L_{Sfer_b}\right)_{Cal} \\
\delta\left[\left(L_{Asym_a} - L_{Asym_b}\right) g + \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\right] \\
= \left[\left(\hat{L}_{Asym_a} - \hat{L}_{Asym_b}\right) g + \left(\hat{L}_{LinAsym_a} - \hat{L}_{LinAsym_b}\right)\right] \\
- \left[\left(L_{Asym_a} - L_{Asym_b}\right) g + \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\right]_{Cal} \\
\delta\left(\theta_{ay} - \theta_{by}\right) = \left(\hat{\theta}_{ay} - \hat{\theta}_{by}\right) - \left(\theta_{ay} - \theta_{by}\right)_{Cal} \\
\delta\left(\theta_{az} - \theta_{bz}\right) = \left(\hat{\theta}_{az} - \hat{\theta}_{bz}\right) - \left(\theta_{az} - \theta_{bz}\right)_{Cal}
\end{gathered}
\tag{9}
$$

where $\delta(\;)$ signifies the residual of parameter ( ) following calibration for assessing the stability of ( ) over test environments, and subscript *Cal* denotes the calibration measurement of the associated parameter determined prior to the start of testing (as described previously).

## CONCLUSIONS

This section concludes the article (except for the appendix and references to follow).

## APPENDIX A

## ANALYTICAL DERIVATIONS

This appendix provides analytical detail leading to Eqs. (6) and (7) in the main text.

First, a detailed expression for accelerometer input $a_{In_i}$ is determined from its definition in (2):

$$
\begin{gathered}
a_{In_i} = \underline{u}_{In}^T \, C_{\theta_a} \, C_{\phi_i} \, \underline{a}_i \qquad \underline{u}_{In}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\
C_{\theta_a} = I + \left(\underline{\theta}_a \times\right) + \frac{1}{2}\left(\underline{\theta}_a \times\right)^2 + \cdots \qquad C_{\phi_i} = I + \left(\underline{\phi}_i \times\right) + \frac{1}{2}\left(\underline{\phi}_i \times\right)^2 + \cdots
\end{gathered}
\tag{A-1}
$$

For each cube $i$ orientation, $a_{In_i}$ in (A-1) is analytically defined in more detail by first deriving an equation for $\underline{u}_{In}^T C_{\theta_a}$, then for $C_{\phi_i} \underline{a}_i$ with $\underline{a}_i$ provided by (2). For $\underline{u}_{In}^T C_{\theta_a}$, using the $(\underline{\theta}_a \times)$ and $\frac{1}{2}(\underline{\theta}_a \times)^2$ formulas in (3) gives

$$\underline{u}_{In}^T C_{\theta_a}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left\{ I + \begin{bmatrix} 0 & -\theta_{az} & \theta_{ay} \\ \theta_{az} & 0 & -\theta_{ax} \\ -\theta_{ay} & \theta_{ax} & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\left(\theta_{az}^2 + \theta_{ay}^2\right) & \theta_{ax}\,\theta_{ay} & \theta_{az}\,\theta_{ax} \\ \theta_{ax}\,\theta_{ay} & -\left(\theta_{ax}^2 + \theta_{az}^2\right) & \theta_{ay}\,\theta_{az} \\ \theta_{az}\,\theta_{ax} & \theta_{ay}\,\theta_{az} & -\left(\theta_{ay}^2 + \theta_{ax}^2\right) \end{bmatrix} \right\} \quad \text{(A-2)}$$

$$= \begin{bmatrix} 1 - 0.5\left(\theta_{az}^2 + \theta_{ay}^2\right) & -\theta_{az} + 0.5\theta_{ax}\,\theta_{ay} & \theta_{ay} + 0.5\theta_{az}\,\theta_{ax} \end{bmatrix}$$

The $C_{\phi_i} \underline{a}_i$ expression in (A-2) is evaluated for each $i$ using the $(\underline{\phi}_i \times)$ and $\frac{1}{2}(\underline{\phi}_i \times)^2$ formulas in (3):

$$C_{\phi_i} \underline{a}_i \approx \left\{ I + \begin{bmatrix} 0 & -\phi_{iz} & \phi_{iy} \\ \phi_{iz} & 0 & -\phi_{ix} \\ -\phi_{iy} & \phi_{ix} & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\left(\phi_{iz}^2 + \phi_{iy}^2\right) & \phi_{ix}\,\phi_{iy} & \phi_{iz}\,\phi_{ix} \\ \phi_{ix}\,\phi_{iy} & -\left(\phi_{ix}^2 + \phi_{iz}^2\right) & \phi_{iy}\,\phi_{iz} \\ \phi_{iz}\,\phi_{ix} & \phi_{iy}\,\phi_{iz} & -\left(\phi_{iy}^2 + \phi_{ix}^2\right) \end{bmatrix} \right\} \underline{a}_i \quad \text{(A-3)}$$

Substituting the $\underline{a}_i$ values in (1), $C_{\phi_i} \underline{a}_i$ in (A-3) is then evaluated for each $i$ orientation. For example, for $i = 1$,

$$C_{\phi_1} \underline{a}_1 \approx \left\{ I + \begin{bmatrix} 0 & -\phi_{1z} & \phi_{1y} \\ \phi_{1z} & 0 & -\phi_{1x} \\ -\phi_{1y} & \phi_{1x} & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\left(\phi_{1z}^2 + \phi_{1y}^2\right) & \phi_{1x}\,\phi_{1y} & \phi_{1z}\,\phi_{1x} \\ \phi_{1x}\,\phi_{1y} & -\left(\phi_{1x}^2 + \phi_{1z}^2\right) & \phi_{1y}\,\phi_{1z} \\ \phi_{1z}\,\phi_{1x} & \phi_{1y}\,\phi_{1z} & -\left(\phi_{1y}^2 + \phi_{1x}^2\right) \end{bmatrix} \right\}_{i=1} \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} \quad \text{(A-4)}$$

$$= \begin{bmatrix} 1 - 0.5\left(\phi_{1z}^2 + \phi_{1y}^2\right) \\ \phi_{1z} + 0.5\,\phi_{1x}\,\phi_{1y} \\ -\phi_{1y} + 0.5\,\phi_{1z}\,\phi_{1x} \end{bmatrix} g$$

For $i = 3$:

$$C_{\phi_3}\,\underline{a}_3 \approx \left\{ I + \begin{bmatrix} 0 & -\phi_{3z} & \phi_{3y} \\ \phi_{3z} & 0 & -\phi_{3x} \\ -\phi_{3y} & \phi_{3x} & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\left(\phi_{3z}^2 + \phi_{3y}^2\right) & \phi_{3x}\,\phi_{3y} & \phi_{3z}\,\phi_{3x} \\ \phi_{3x}\,\phi_{2y} & -\left(\phi_{3x}^2 + \phi_{3z}^2\right) & \phi_{3y}\,\phi_{3z} \\ \phi_{3z}\,\phi_{3x} & \phi_{3y}\,\phi_{3z} & -\left(\phi_{3y}^2 + \phi_{3x}^2\right) \end{bmatrix} \right\} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

$$= -\begin{bmatrix} \phi_{3y} + 0.5\phi_{3z}\,\phi_{3x} \\ -\phi_{3x} + 0.5\,\phi_{3y}\,\phi_{3z} \\ 1 - 0.5\left(\phi_{3y}^2 + \phi_{3x}^2\right) \end{bmatrix} g \qquad \text{(A-5)}$$

and similarly for $C_{\phi_4}\,\underline{a}_4$. With $C_{\phi_1}\,\underline{a}_1$, $C_{\phi_3}\,\underline{a}_3$, $C_{\phi_4}\,\underline{a}_4$ so found, $C_{\phi_2}\,\underline{a}_2$, $C_{\phi_5}\,\underline{a}_5$, $C_{\phi_6}\,\underline{a}_6$ are easily obtained by noting from (1) and (A-1) that

$$\begin{aligned} &C_{\phi_2}\,\underline{a}_2 = -\,C_{\phi_1}\,\underline{a}_1 \text{ with 1 replaced by 2} \\ &C_{\phi_5}\,\underline{a}_5 = -\,C_{\phi_3}\,\underline{a}_3 \text{ with 3 replaced by 5} \\ &C_{\phi_6}\,\underline{a}_6 = -\,C_{\phi_4}\,\underline{a}_4 \text{ with 4 replaced by 4} \end{aligned} \qquad \text{(A-6)}$$

Having derived $\underline{u}_{In}^T\,C_{\theta_a}$ and $C_{\phi_i}\,\underline{a}_i$ for each *i*, applying them in (A-2) then provides the complete input acceleration set:

$$\begin{aligned} a_{In_1} &\approx \left[1 - 0.5\left(\theta_{az}^2 + \theta_{ay}^2 + \phi_{1z}^2 + \phi_{1y}^2\right) - \theta_{az}\phi_{1z} - \theta_{ay}\phi_{1y}\right] g \\ a_{In_2} &\approx -\left[1 - 0.5\left(\theta_{az}^2 + \theta_{ay}^2 + \phi_{2z}^2 + \phi_{2y}^2\right) - \theta_{az}\phi_{2z} - \theta_{ay}\phi_{2y}\right] g \\ a_{In_3} &\approx -\left[\phi_{3y} + 0.5\phi_{3z}\,\phi_{3x} + \theta_{az}\,\phi_{3x} + \theta_{ay} + 0.5\theta_{az}\,\theta_{ax}\right] g \\ a_{In_5} &\approx \left[\phi_{5y} + 0.5\phi_{5z}\,\phi_{5x} + \theta_{az}\,\phi_{5x} + \theta_{ay} + 0.5\theta_{az}\,\theta_{ax}\right] g \\ a_{In_4} &\approx -\left[-\phi_{4z} + 0.5\phi_{4x}\,\phi_{4y} - \theta_{az} + 0.5\theta_{ax}\,\theta_{ay} + \theta_{ay}\phi_{4x}\right] g \\ a_{In_6} &\approx \left[-\phi_{6z} + 0.5\phi_{6x}\,\phi_{6y} - \theta_{az} + 0.5\theta_{ax}\,\theta_{ay} + \theta_{ay}\phi_{6x}\right] g \end{aligned} \qquad \text{(A-7)}$$

The output from accelerometer *a* is found by applying (A-7) to the Eq. (2) output equation:

$$a_{Out_{ai}} = a_{Bias_a} + \left(1 + L_{Sfer_a} + L_{Asym_a}\,a_{In_i} + L_{LinAsym_a}\,sign(a_{In_i})\right) a_{In_i} \qquad \text{(A-8)}$$

Note in (A-8) that $x\;sign(x) = |x|$ for any x. Substituting (A-7) in (A-8) while applying the previous note, obtains the accelerometer output equation for each *i* orientation. For *i* = 1,

$$a_{Out_{a1}} = a_{Bias_a} + \left(1 + L_{Sfer_a} + L_{Asym_a}\, a_{In1} + L_{LinAsym_a}\, sign(a_{In1})\right) a_{In1}$$

$$\approx a_{Bias_a} + \begin{pmatrix} 1 + L_{Sfer_a} + L_{Asym_a}\, a_{In1} \\ + L_{LinAsym_a}\, sign(a_{In1}) \end{pmatrix} \left[1 - 0.5\left(\theta_{az}^2 + \theta_{ay}^2 + \phi_{1z}^2 + \phi_{1y}^2\right) - \theta_{az}\phi_{1z} - \theta_{ay}\phi_{1y}\right] g \qquad \text{(A-9)}$$

$$\approx a_{Bias_a} + \begin{bmatrix} 1 - 0.5\left(\theta_{az}^2 + \theta_{ay}^2 + \phi_{1z}^2 + \phi_{1y}^2\right) - \theta_{az}\phi_{1z} - \theta_{ay}\phi_{1y} \\ + L_{Sfer_a} + L_{Asym_a}\, g + L_{LinAsym_a} \end{bmatrix} g$$

and for $i = 3$:

$$a_{Out_{a3}} = a_{Bias_a} + \left(1 + L_{Sfer_a} + L_{Asym_a}\, a_{In3} + L_{LinAsym_a}\, sign(a_{In3})\right) a_{In3}$$

$$\approx a_{Bias_a} - \begin{bmatrix} 1 + L_{Sfer_a} - L_{Asym_a}\left(-\phi_{3y}\, g\right) \\ + L_{LinAsym_a}\, sign(-\phi_{3y}) \end{bmatrix} \left[\phi_{3y} + 0.5\phi_{3z}\,\phi_{3x} + \theta_{az}\,\phi_{3x} + \theta_{ay} + 0.5\theta_{az}\,\theta_{ax}\right] g$$

$$\approx a_{Bias_a} - \begin{bmatrix} \phi_{3y} + 0.5\phi_{3z}\,\phi_{3x} + \theta_{az}\,\phi_{3x} + \theta_{ay} + 0.5\theta_{az}\,\theta_{ax} \\ + \left(L_{Sfer_a} + L_{LinAsym_a}\, sign(-\phi_{3y})\right)\, \phi_{3y} \end{bmatrix} g \qquad \text{(A-10)}$$

$$= a_{Bias_a} - \begin{bmatrix} \phi_{3y} + 0.5\phi_{3z}\,\phi_{3x} + \theta_{az}\,\phi_{3x} + \theta_{ay} + 0.5\theta_{az}\,\theta_{ax} \\ + L_{Sfer_a}\, \phi_{3y} - L_{LinAsym_a} \left|\, \phi_{3y}\right| \end{bmatrix} g$$

The complete set of accelerometer outputs thereby becomes

$$\begin{aligned}
a_{Out_{a1}} &\approx a_{Bias_a} + \begin{bmatrix} 1 - 0.5\left(\theta_{az}^2 + \theta_{ay}^2 + \phi_{1z}^2 + \phi_{1y}^2\right) - \theta_{az}\phi_{1z} - \theta_{ay}\phi_{1y} \\ + L_{Sfer_a} + L_{Asym_a}\, g + L_{LinAsym_a} \end{bmatrix} g \\
a_{Out_{a2}} &\approx a_{Bias_a} - \begin{bmatrix} 1 - 0.5\left(\theta_{az}^2 + \theta_{ay}^2 + \phi_{2z}^2 + \phi_{2y}^2\right) - \theta_{az}\phi_{2z} - \theta_{ay}\phi_{2y} \\ + L_{Sfer_a} - L_{Asym_a}\, g - L_{LinAsym_a} \end{bmatrix} g \\
a_{Out_{a3}} &\approx a_{Bias_a} - \begin{bmatrix} \phi_{3y} + 0.5\phi_{3z}\,\phi_{3x} + \theta_{az}\,\phi_{3x} + \theta_{ay} + 0.5\theta_{az}\,\theta_{ax} \\ + L_{Sfer_a}\, \phi_{3y} - L_{LinAsym_a} \left|\, \phi_{3y}\right| \end{bmatrix} g \\
a_{Out_{a5}} &\approx a_{Bias_a} + \begin{bmatrix} \phi_{5y} + 0.5\phi_{5z}\,\phi_{5x} + \theta_{az}\,\phi_{5x} + \theta_{ay} + 0.5\theta_{az}\,\theta_{ax} \\ + L_{Sfer_a}\, \phi_{5y} + L_{LinAsym_a} \left|\, \phi_{5y}\right| \end{bmatrix} g \qquad \text{(A-11)} \\
a_{Out_{a4}} &\approx a_{Bias_a} - \begin{bmatrix} -\phi_{4z} + 0.5\phi_{4x}\,\phi_{4y} - \theta_{az} + 0.5\theta_{ax}\,\theta_{ay} + \theta_{ay}\phi_{4x} \\ - L_{Sfer_a}\, \phi_{4z} - L_{LinAsym_a} \left|\phi_{4z}\right| \end{bmatrix} g \\
a_{Out_{a6}} &\approx a_{Bias_a} + \begin{bmatrix} -\phi_{6z} + 0.5\phi_{6x}\,\phi_{6y} - \theta_{az} + 0.5\theta_{ax}\,\theta_{ay} + \theta_{ay}\phi_{6x} \\ - L_{Sfer_a}\, \phi_{6z} + L_{LinAsym_a} \left|\phi_{6z}\right| \end{bmatrix} g
\end{aligned}$$

An equivalent to (A-11) for the *b* accelerometer is easily generated by replacing *a* with *b*.

Having equations for the accelerometer *a* outputs in (A-11) and their accelerometer *b* equivalents, the Z measurement for each *i* orientation is found by applying them in (5). For example, for $i = 1$,

$$
\begin{aligned}
Z_1 = a_{Out_{a1}} - a_{Out_{b1}} = a_{Bias_a} &+ \begin{bmatrix} 1 - 0.5\left(\theta_{az}^2 + \theta_{ay}^2 + \phi_{1z}^2 + \phi_{1y}^2\right) - \theta_{az}\phi_{1z} - \theta_{ay}\phi_{1y} \\ + L_{Sfer_a} + L_{Asym_a}\, g + L_{LinAsym_a} \end{bmatrix} g \\
- a_{Bias_b} &- \begin{bmatrix} 1 - 0.5\left(\theta_{bz}^2 + \theta_{by}^2 + \phi_{1z}^2 + \phi_{1y}^2\right) - \theta_{bz}\phi_{1z} - \theta_{by}\phi_{1y} \\ + L_{Sfer_b} + L_{Asym_b}\, g + L_{LinAsym_b} \end{bmatrix} g \\
= \left(a_{Bias_a} - a_{Bias_b}\right) &+ \begin{bmatrix} -0.5\left(\theta_{az}^2 + \theta_{ay}^2 + \phi_{1z}^2 + \phi_{1y}^2 - \theta_{bz}^2 - \theta_{by}^2 - \phi_{1z}^2 - \phi_{1y}^2\right) \\ -\theta_{az}\phi_{1z} - \theta_{ay}\phi_{1y} + L_{Sfer_a} + L_{Asym_a}\, g + L_{LinAsym_a} \\ +\theta_{bz}\phi_{1z} + \theta_{by}\phi_{1y} - L_{Sfer_b} - L_{Asym_b}\, g - L_{LinAsym_b} \end{bmatrix} g \qquad \text{(A-12)} \\
= \left(a_{Bias_a} - a_{Bias_b}\right) &+ \begin{bmatrix} -0.5\left[\left(\theta_{az}^2 - \theta_{bz}^2\right) + \left(\theta_{ay}^2 - \theta_{by}^2\right)\right] \\ -(\theta_{az} - \theta_{bz})\phi_{1z} - \left(\theta_{ay} - \theta_{by}\right)\phi_{1y} + \left(L_{Sfer_a} - L_{Sfer_b}\right) \\ +\left(L_{Asym_a} - L_{Asym_b}\right) g + \left(L_{LinAsym_a} - L_{LinAsym_b}\right) \end{bmatrix} g
\end{aligned}
$$

The complete Z set is thereby found to be

$$
\begin{aligned}
Z_1 &= \left(a_{Bias_a} - a_{Bias_b}\right) + \begin{bmatrix} -0.5\left[\left(\theta_{az}^2 - \theta_{bz}^2\right) + \left(\theta_{ay}^2 - \theta_{by}^2\right)\right] \\ -(\theta_{az} - \theta_{bz})\phi_{1z} - \left(\theta_{ay} - \theta_{by}\right)\phi_{1y} + \left(L_{Sfer_a} - L_{Sfer_b}\right) \\ +\left(L_{Asym_a} - L_{Asym_b}\right) g + \left(L_{LinAsym_a} - L_{LinAsym_b}\right) \end{bmatrix} g \\
Z_2 &= \left(a_{Bias_a} - a_{Bias_b}\right) - \begin{bmatrix} -0.5\left[\left(\theta_{az}^2 - \theta_{bz}^2\right) + \left(\theta_{ay}^2 - \theta_{by}^2\right)\right] \\ -(\theta_{az} - \theta_{bz})\phi_{2z} - \left(\theta_{ay} - \theta_{by}\right)\phi_{2y} + \left(L_{Sfer_a} - L_{Sfer_b}\right) \\ -\left(L_{Asym_a} - L_{Asym_b}\right) g - \left(L_{LinAsym_a} - L_{LinAsym_b}\right) \end{bmatrix} g
\end{aligned} \qquad \text{(A-13)}
$$

(Continued)

[(A-13) Concluded]

$$Z_3 = \left(a_{Bias_a} - a_{Bias_b}\right) - \begin{bmatrix} \left(\theta_{ay} - \theta_{by}\right) + \left(\theta_{az} - \theta_{bz}\right)\phi_{3x} + 0.5\left(\theta_{az}\,\theta_{ax} - \theta_{bz}\,\theta_{bx}\right) \\ +\left(L_{Sfer_a} - L_{Sfer_b}\right)\,\phi_{3y} - \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left|\,\phi_{3y}\right| \end{bmatrix} g$$

$$Z_5 = \left(a_{Bias_a} - a_{Bias_b}\right) + \begin{bmatrix} \left(\theta_{ay} - \theta_{by}\right) + \left(\theta_{az} - \theta_{bz}\right)\phi_{5x} + 0.5\left(\theta_{az}\,\theta_{ax} - \theta_{bz}\,\theta_{bx}\right) \\ +\left(L_{Sfer_a} - L_{Sfer_b}\right)\,\phi_{5y} + \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left|\,\phi_{5y}\right| \end{bmatrix} g$$

$$Z_4 = \left(a_{Bias_a} - a_{Bias_b}\right) - \begin{bmatrix} -\left(\theta_{az} - \theta_{bz}\right) + \left(\theta_{ay} - \theta_{by}\right)\phi_{4x} + 0.5\left(\theta_{ax}\,\theta_{ay} - \theta_{bx}\,\theta_{by}\right) \\ -\left(L_{Sfer_a} - L_{Sfer_b}\right)\,\phi_{4z} - \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left|\phi_{4z}\right| \end{bmatrix} g$$

$$Z_6 = \left(a_{Bias_a} - a_{Bias_b}\right) + \begin{bmatrix} -\left(\theta_{az} - \theta_{bz}\right) + \left(\theta_{ay} - \theta_{by}\right)\phi_{6x} + 0.5\left(\theta_{ax}\,\theta_{ay} - \theta_{bx}\,\theta_{by}\right) \\ -\left(L_{Sfer_a} - L_{Sfer_b}\right)\,\phi_{6z} + \left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left|\phi_{6z}\right| \end{bmatrix} g$$

Eqs. (A-13) shows that each Z measurement is a simple function accelerometer *a* and *b* error differences (to first order) plus a group of second order terms. Simple linear combinations of appropriate Zs allows each to be equated to each sensor error difference (to first order) plus second order terms. Rearrangement then equates each error source difference to its corresponding Z formula plus residual second order terms. For example, the Z combination of $\left(Z_5 + Z_3\right) / 2g$ obtains from (A-13):

$$\begin{aligned} \left(Z_5 + Z_3\right) / 2g &= \left(a_{Bias_a} - a_{Bias_b}\right) / g + 0.5\left(\theta_{az} - \theta_{bz}\right)\left(\phi_{5x} - \phi_{3x}\right) \\ &+0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)\left(\phi_{5y} - \phi_{3y}\right) + 0.5\left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left(\left|\phi_{5y}\right| + \left|\phi_{3y}\right|\right) \end{aligned} \qquad \text{(A-14)}$$

which after rearrangement gives

$$\begin{aligned} \left(a_{Bias_a} - a_{Bias_b}\right) / g &= \left(Z_5 + Z_3\right) / 2g - 0.5\left(\theta_{az} - \theta_{bz}\right)\left(\phi_{5x} - \phi_{3x}\right) \\ &-0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)\left(\phi_{5y} - \phi_{3y}\right) - 0.5\left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left(\left|\phi_{5y}\right| + \left|\phi_{3y}\right|\right) \end{aligned} \qquad \text{(A-15)}$$

Then, for all of the appropriate Z combinations in (A-13):

$$\left(a_{Bias_a} - a_{Bias_b}\right) / g = (Z_5 + Z_3) / 2g - 0.5(\theta_{az} - \theta_{bz})(\phi_{5x} - \phi_{3x})$$
$$- 0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)\left(\phi_{5y} - \phi_{3y}\right) - 0.5\left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left(\left|\phi_{5y}\right| + \left|\phi_{3y}\right|\right)$$

$$\left(L_{Sfer_a} - L_{Sfer_b}\right) = (Z_1 - Z_2) / 2g + 0.5(\theta_{az} - \theta_{bz})(\phi_{1z} - \phi_{2z}) + 0.5\left(\theta_{ay} - \theta_{by}\right)\left(\phi_{1y} - \phi_{2y}\right)$$

$$\left(L_{Asym_a} - L_{Asym_b}\right) g + \left(L_{LinAsym_a} - L_{LinAsym_b}\right) = (Z_1 + Z_2 - Z_5 - Z_3) / 2g$$
$$+ 0.5(\theta_{az} - \theta_{bz})\left(\phi_{1z} - \phi_{2z} + \phi_{5x} - \phi_{3x}\right) + 0.5\left(\theta_{ay} - \theta_{by}\right)\left(\phi_{1y} - \phi_{2y}\right) \qquad \text{(A-16)}$$
$$+ 0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)\left(\phi_{5y} - \phi_{3y}\right) + 0.5\left(L_{LinAsym_a} - L_{LinAsym_b}\right)\left(\left|\phi_{5y}\right| + \left|\phi_{3y}\right|\right)$$

$$\left(\theta_{ay} - \theta_{by}\right) = (Z_5 - Z_3) / 2g - 0.5(\theta_{az} - \theta_{bz})\left(\phi_{5x} + \phi_{3x}\right) - 0.5(\theta_{az}\,\theta_{ax} - \theta_{bz}\,\theta_{bx})$$
$$- 0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)\left(\phi_{5y} + \phi_{3y}\right)$$

$$(\theta_{az} - \theta_{bz}) = (Z_4 - Z_6) / 2g + 0.5\left(\theta_{ay} - \theta_{by}\right)(\phi_{4x} + \phi_{6x}) + 0.5\left(\theta_{ax}\,\theta_{ay} - \theta_{bx}\,\theta_{by}\right)$$
$$- 0.5\left(L_{Sfer_a} - L_{Sfer_b}\right)(\phi_{4z} + \phi_{6x})$$

By neglecting higher order terms, (A-16) simplifies to

$$\left(a_{Bias_a} - a_{Bias_b}\right) / g \approx (Z_5 + Z_3) / 2g$$
$$\left(L_{Sfer_a} - L_{Sfer_b}\right) \approx (Z_1 - Z_2) / 2g$$
$$\left(L_{Asym_a} - L_{Asym_b}\right) g + \left(L_{LinAsym_a} - L_{LinAsym_b}\right) \approx (Z_1 + Z_2 - Z_5 - Z_3) / 2\text{g} \qquad \text{(A-17)}$$
$$\left(\theta_{ay} - \theta_{by}\right) \approx (Z_5 - Z_3) / 2g \qquad (\theta_{az} - \theta_{bz}) \approx (Z_4 - Z_6) / 2g$$

Eq. (A-17) is the basis for Eq. (8) in the main text.

REFERENCES

[1] Savage, P. G., *Strapdown Analytics, Edition II*, Strapdown Associates, Inc., 2007, available for purchase at www.strapdownassociates.com.