MOVING BASE INS ALIGNMENT WITH LARGE INITIAL HEADING ERROR

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ABSTRACT

Prior to inertial navigation mode engagement, an inertial navigation system (INS) executes Kalman filter attitude alignment operations, an estimation process based on minimizing a linearized model of INS attitude errors. Kalman error model linearization places a constraint on INS initial attitude before initiating Kalman alignment operations: attitude errors at the start of Kalman alignment must be small enough that residual second order error effects from linearized modeling have negligible impact on alignment accuracy. Under stationary alignment conditions, linearized Kalman error models have been commonly configured for large initial heading error with no second order heading induced errors, by representing heading error as inaccuracy in estimating horizontal earth rate components. For non-stationary dynamic alignment conditions, however, the earth rate estimation alignment approach cannot be used directly. The difficulty arises when forming the Kalman filter measurement input; a navigation data comparison between the INS and the equivalent data provided by a reference navigation device (e.g., GPS derived velocity). The data measurement comparison must be made in a common coordinate frame which becomes problematic when INS heading is initially unknown. Based on a recent INS alignment article for small initial heading error, this article shows how the Kalman filter measurement can be formulated so that large initial INS heading uncertainty generates negligible second order error impact on alignment accuracy under dynamic motion conditions.

MATHEMATICAL NOTATION

$\underline{V}$ = Vector without specific coordinate frame designation. A vector is a parameter that has length and direction. Vectors used in the paper are classified as "free vectors", hence, have no preferred location in coordinate frames in which they are analytically described.

$|\underline{V}|$ or V = Magnitude of vector $\underline{V}$.

$\underline{V}^A$ = Column matrix with elements equal to the projection of $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit vector parallel to that coordinate axis.

$\left(\underline{V}^{A}\times\right)$ = Skew symmetric (or cross-product) form of $\underline{V}^{A}$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{XA}, V_{YA}, V_{ZA} are the components of $\underline{V}^{A}$. The matrix product of $\left(\underline{V}^{A}\times\right)$ with another A frame vector equals the cross-product of $\underline{V}^{A}$ with the vector in the A frame, i.e.: $\left(\underline{V}^{A}\times\right)\underline{W}^{A} = \underline{V}^{A}\times\underline{W}^{A}$.

$C_{A_2}^{A_1}$ = Direction cosine matrix that transforms a vector from its coordinate frame A_2 projection form to its coordinate frame A_1 projection form, i.e.: $\underline{V}^{A_1} = C_{A_2}^{A_1}\,\underline{V}^{A_2}$. The columns of $C_{A_2}^{A_1}$ are projections on A_1 axes of unit vectors parallel to A_2 axes. Conversely, the rows of $C_{A_2}^{A_1}$ are projections on A_2 axes of unit vectors parallel to A_1 axes. An important property of $C_{A_2}^{A_1}$ is that it's inverse equals it's transpose.

$\underline{\omega}_{A_1 A_2}$ = Angular rotation rate of coordinate frame A_2 relative to coordinate frame A_1. Conversely, the angular rotation rate of coordinate frame A_1 relative to coordinate frame A_2 is the negative of $\underline{\omega}_{A_1 A_2}$, i.e.,: $\underline{\omega}_{A_2 A_1} = -\underline{\omega}_{A_1 A_2}$.

$\dot{(\ )} = \dfrac{d(\ )}{dt}$ = Derivative with respect to time t.

$\widehat{(\ )}$ = Computed value of parameter ( ) that, in contrast with the idealized error free value ( ), contains errors.

$\widetilde{(\ )}$ = Measured value of strapdown inertial sensor ( ) that, in contrast with the idealized error free value ( ), contains errors.

$\delta(\ )$ = Designation for errors that are small compared with ( ).

$\Delta(\ )$ = Designation for errors that can be as large as ( ).

$\mathcal{E}$ = Expected value operator.

COORDINATE FRAMES

N = INS locally level navigation coordinate frame (with Z axis up) used for attitude referencing and velocity/position integration operations. By definition in this article, the initial heading of the N Frame is assumed to be nominal, i.e., error free. Initial heading alignment of the N Frame relative to another known reference frame (N*) is accounted for by defining the N frame to be nominally misaligned from the N* frame.

N* = Locally level navigation coordinate frame (with Z axis up) used by a reference aiding device to deliver position/velocity data to the INS being aligned. The heading angle misalignment between the N and N* frames is the means to account for initial heading error in the INS attitude data at the start of alignment. As defined, the Z axis of the N* frame is parallel to the Z axis of the N Frame.

B = Strapdown inertial sensor coordinates ("body frame") with axes parallel to nominal right handed orthogonal sensor input axes.

I = Non-rotating inertial coordinate frame used as a symbolic reference for gyro angular rotation rate measurements.

E = Coordinate (earth) frame aligned with axes fixed to the earth.

PARAMETER DEFINITIONS

Analytic parameters used in this article are defined following equations where they are first used.

INTRODUCTION

An important part of inertial navigation system (INS) operations is the initialization process in which the INS navigation parameters (angular orientation - attitude, velocity, and position) are initialized for integration functions to follow during inertial navigation. Initialization typically consists of two phases; Coarse Alignment followed by Fine Alignment, e.g., [3 - Chapt. 6]. During Coarse Alignment, the INS angular attitude is initialized to an approximately correct value. Fine Alignment then ensues in which attitude is converged to inertial navigation grade accuracy and velocity/position are initialized, all generally implemented within a Kalman filter structure.

General Inertial Aiding Structure

Fine Alignment is a specialized application of Inertial Navigation Aiding (Figure 1), a dynamic process in which INS computed navigation data is periodically compared with equivalent reference navigation data (at cycle rate n), and used in feedback fashion to update INS error parameters. Note in Figure 1 that all parameters are shown with a $\widehat{( )}$ designation to indicate that they are computed estimates within the INS and reference navigation device of actual equivalent ( ) parameters. The analytic details of the Figure 1 operations are provided in [1, 2, 3 - Chapt. 15, 4 - pp. 415 - 457].

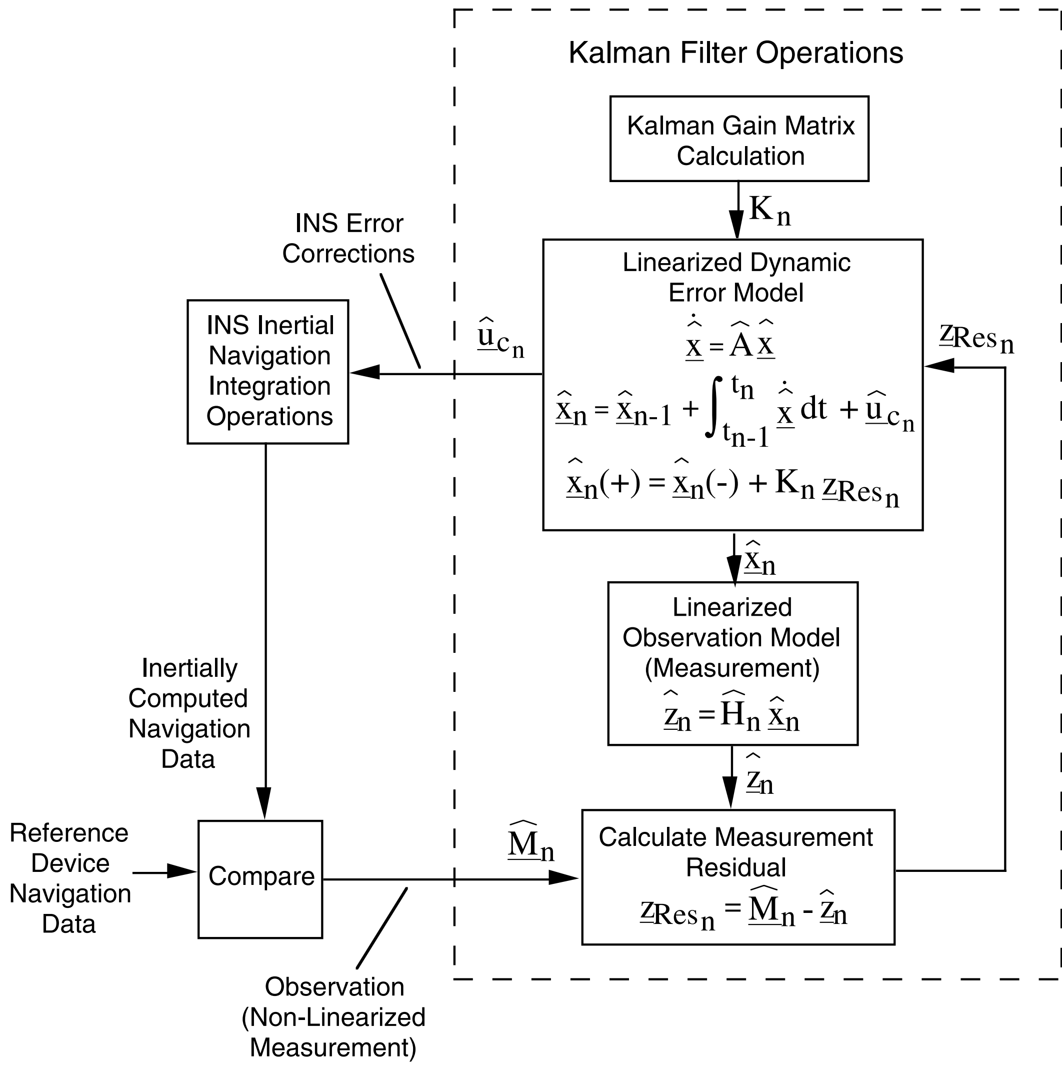

Figure 1 - Inertial Navigation Aiding

In Figure 1, the inertially-computed/reference-device navigation data comparison $\underline{\widehat{M}}$ ("observation") is input to the Kalman filter where it is compared against a linearized estimate of $\underline{\widehat{M}}$, known as the "measurement" $\underline{\hat{z}}$. The equation for $\underline{\hat{z}}$ is based on estimates of expected errors (embodied in an error state vector column matrix $\underline{\hat{x}}$) generated by a linearized dynamic error model of inertial navigation and reference device operations, and how they couple into the measurement (through the "measurement matrix" $\hat{H}$). (The error state dynamic matrix $\hat{A}$ in Figure 1 defines the dynamics of how $\underline{\hat{x}}$ propagate from the last n cycle to the current n cycle.) The difference between the observation $\widehat{\underline{M}}$ and estimated measurement $\hat{\underline{z}}$ (the "measurement residual" $\underline{z}_{Res}$), is multiplied by a Kalman gain matrix K_n to generate corrections to the Kalman filter error estimates. The control vector $\hat{\underline{u}}_c$ formed from INS error estimates in $\hat{\underline{x}}$ (including provisions for $\hat{\underline{x}}$ computation delay) is used to correct the INS by subtraction from the equivalent INS parameter data. To account for the $\hat{\underline{u}}_c$ corrections applied to the INS, the $\hat{\underline{u}}_c$ vector is also used to update the Kalman $\hat{\underline{x}}$ error model for the applied INS error correction.

The K_n Kalman gain matrix in Figure 1 is computed at each n cycle from a statistical model of the expected uncertainty in the Figure 1 linearized updating process, a function of the error state covariance matrix P:

$$
\begin{gathered}
P_n = P_{n-1} + \int_{t_{n-1}}^{t_n} \dot{P}\, dt \qquad \dot{P} = \hat{A}\, P + P\, \hat{A}^T + \hat{G}_P\, Q_P\, \hat{G}_P^T \\
K_n = P_n(-)\, \hat{H}_n^T \left(\hat{H}_n\, P_n(-)\, \hat{H}_n^T + \hat{G}_{M_n}\, R_M\, \hat{G}_{M_n}^T\right)^{-1} \qquad (1) \\
P_n(+) = \left(I - K_n\, \hat{H}_n\right) P_n(-) \left(I - K_n\, \hat{H}_n\right)^T + K_n\, \hat{G}_{M_n}\, R_M\, \hat{G}_{M_n}^T\, K_n^T
\end{gathered}
$$

The P covariance is analytically defined as $\mathcal{E}\left(\underline{x}_{Uncrnty}\, \underline{x}_{Uncrnty}^T\right)$ where $\mathcal{E}$ is the expected value operator and $\underline{x}_{Uncrnty}$ is the uncertainty in the error state estimate $\hat{\underline{x}}$ compared with the true value $\underline{x}$. The covariance matrix measures how initial uncertainties in $\hat{\underline{x}}$ (at the start of Fine Alignment) are progressively reduced by the Figure 1 dynamic estimation/updating process, and how unaccounted for noise effects (in $\hat{\underline{x}}$ propagation between updates and measurement updating) delay the convergence process. Noise parameters incorporated in the (1) gain determination operations are the Q_P process noise matrix that accounts for random INS error buildup between n cycles, the $\hat{G}_P$ matrix that couples the process noise into error state uncertainty components, the measurement noise matrix R_M that accounts for random errors in the observation and in calculation of the measurement residual, and the $\hat{G}_M$ matrix that couples the measurement noise into the measurement residual components [3 - Sect. 15.1] and [4 - pp. 428].

The success of the Figure 1 process depends on the accuracy by which the Kalman filter linearized models match the actual operations in the INS and reference navigation device. An important element in this regard is the impact of linearization on the measurement residual. Second order components in the Figure 1 observation vector $\underline{M}$ are ignored in the Kalman filter linearized models, hence, will appear in the measurement residual $\hat{\underline{z}}_{Res}$ and modify $\underline{x}$ through the Kalman gains. Since the gains do not account for second order errors, the result will add unknown errors to $\hat{\underline{x}}$. To minimize the impact of second order errors on Kalman filter performance, it has been previous practice to assure that Coarse Alignment attitude errors are small enough that second order residuals become negligible. However, in some applications, residual second order Kalman filter modeling errors can still produce mis-estimation of INS errors under particular dynamic conditions [5, 6].

Application of Kalman Aiding To Velocity Matching INS Alignment

Applications of the Figure 1 inertial aiding concept to INS Fine Alignment commonly use velocity for the INS/reference-device data comparison (so-called "velocity matching" alignment). The associated inertial navigation integration operations in Figure 1 (between Kalman filter applied $\hat{\underline{u}}_c$ control updates) are versions of the following:

$$\hat{C}^N_{B_n} = \hat{C}^N_{B_{n-1}} + \int_{t_{n-1}}^{t_n} \dot{\hat{C}}^N_B \, dt \qquad \dot{\hat{C}}^N_B = \hat{C}^N_B \left(\underline{\tilde{\omega}}^B_{IB} \times \right) - \underline{\hat{\omega}}^N_{IN} \, \hat{C}^N_B$$

$$\underline{\hat{\omega}}^N_{IN} = \underline{\hat{\omega}}^N_{IE} + \underline{\hat{\omega}}^N_{EN} \tag{2}$$

$$\underline{\hat{v}}^N_n = \underline{\hat{v}}^N_{n-1} + \int_{t_{n-1}}^{t_n} \dot{\underline{\hat{v}}}^N \, dt \qquad \dot{\underline{\hat{v}}}^N = \hat{C}^N_B \, \underline{\tilde{a}}^B_{SF} + \underline{\hat{g}}^N_P - \left(\underline{\hat{\omega}}^N_{IN} + \underline{\hat{\omega}}^N_{IE} \right) \times \underline{\hat{v}}^N$$

where

n = Subscript indicating value of the designated parameter at the Kalman filter update cycle time.

$\underline{\omega}^B_{IB}$ = Angular rate of the strapdown inertial sensor B frame relative to the non-rotating inertial I frame (IB subscript) measured in the B frame (superscript) by INS gyros

C^N_B = Direction cosine matrix that transforms vectors from the sensor body B frame to the locally level navigation N frame.

$\underline{\omega}^N_{IE}$ = Angular rate of the earth fixed E frame relative to the non-rotating I frame (IE) projected on N frame axes.

$\underline{\omega}^N_{EN}$ = Angular rate of the N frame relative to the E frame (EN) projected on N frame axes.

$\underline{v}^N$ = Velocity vector relative to the earth projected on N Frame axes.

$\underline{a}^B_{SF}$ = Specific force (non-gravitational) acceleration measured in the strapdown sensor B frame by INS accelerometers.

$\underline{g}_P^N$ = Plumb-bob gravity in the N frame that equals the sum of earth's gravitational mass attraction plus earth's rotation centripetal acceleration effect. Defined as such because $\underline{g}_P$ lies along the direction of a plumb-bob under zero velocity conditions.

Under dynamic velocity matching trajectory conditions, [5] shows that residual second order Kalman filter modeling errors can produce mis-estimation of small initial heading errors under mild maneuvering conditions. Reference [5] shows how the second order error effects can be mitigated as part of Kalman gain matrix determination by modeling them into the Figure 1 measurement and process noise structure (within the Equation (1) $\hat{Q}_P$, $\hat{G}_P$, $\hat{R}_M$ and $\hat{G}_M$ matrices). As an alternate, [6] describes a Fine Alignment approach whereby second order errors can be minimized by using a modified form of the traditional velocity matching observation equation.

Under the special case of quasi-stationary conditions (e.g., stationary ground alignment of an INS in an aircraft), [3 - Sects. 6.1.2 & 15.2.1] shows how a velocity matching observation can be structured to make large initial heading errors observable within a linearized structure that has no significant second-order errors. The concept is based on (2) under zero velocity conditions:

$$
\begin{aligned}
&\hat{C}^N_{B_n} = \hat{C}^N_{B_{n-1}} + \int_{t_{n-1}}^{t_n} \dot{\hat{C}}^N_B \, dt \qquad \dot{\hat{C}}^N_B = \hat{C}^N_B \left(\underline{\tilde{\omega}}^B_{IB} \times\right) - \left(\underline{\hat{\omega}}^N_{IE} \times\right) \hat{C}^N_B \\
&\underline{\hat{v}}^N_{H_n} = \underline{\hat{v}}^N_{H_{n-1}} + \int_{t_{n-1}}^{t_n} \dot{\underline{\hat{v}}}^N_H \, dt \qquad \underline{\hat{v}}^N_{H_0} = 0 \qquad \dot{\underline{\hat{v}}}^N_H = - \hat{g}_{P_{ZN}} \left(\hat{C}^N_B \, \underline{u}^N_{ZN}\right)_H
\end{aligned}
\tag{3}
$$

where

H = Subscript indicating horizontal components of the designated vector.

$\underline{u}^N_{ZN}$ = Unit vector along the upward N Frame Z axis (ZN).

$\hat{g}_{P_{ZN}}$ = Component of computed plumb-bob gravity $\underline{\hat{g}}^N_P$ along the N frame upward Z axis (of negative sign).

In forming (3) from (2) it has been recognized that under stationary conditions, the initial value of velocity $\underline{\hat{v}}^N_{0_H}$ is zero, $\underline{\tilde{a}}^B_{SF}$ in (2) is vertical to balance plumb-bob gravity, and $\underline{\hat{\omega}}^N_{EN}$ in (2) is proportional to velocity, hence, not present in (3). The computed $\underline{\hat{v}}^N_H$ in (3) then becomes the Figure 1 observation (i.e., compared with the true zero value). The (3) form allows the initial N frame to be nominally defined at any arbitrary heading, then used during Fine Alignment to estimate the N frame horizontal components of earth rate $\underline{\hat{\omega}}^N_{IE}$. Using the principle that horizontal earth rate points north, the estimated earth rate components are then used at Fine Alignment completion to determine the orientation of the selected N Frame relative to true north.

Under moving-base INS alignment conditions, the (3) simplified quasi-stationary earth rate estimation concept cannot be implemented directly because of the non-zero velocity terms in the general (2) form. Although the magnitude of initial velocity $\hat{\underline{v}}_0^N$ in (2) is known (from reference velocity input data magnitude), its vector direction has uncertainty in the N frame because of the initially unknown heading angle between the INS N frame and the reference data input data coordinate frame (identified herein as N*). The $\hat{\underline{v}}_0^N$ error in (2) thereby becomes part of the inertially computed velocity $\hat{\underline{v}}^N$, generating error in $\hat{\underline{\omega}}_{EN}^N$ (a function of $\hat{\underline{v}}^N$), in $\hat{C}_B^N$ from $\hat{\underline{\omega}}_{EN}^N$, in adding to $\hat{\underline{v}}^N$ error through $\hat{C}_B^N$ coupling of $\tilde{\underline{a}}_{SF}^B$ into $\dot{\hat{\underline{v}}}^N$, and by $\hat{\underline{v}}^N$ input in (2) to the $\dot{\hat{\underline{v}}}^N$ Coriolis acceleration (the last term in $\dot{\hat{\underline{v}}}^N$). The traditional method for implementing (2) has been to allow for a heading error component in $\hat{C}_B^N$ for estimation in the Kalman filter linearized error model (in addition to $\hat{C}_B^N$ tilt misalignment from vertical), thereby leading to the requirement for small initial tilt/heading errors (to minimize residual second order error effects). In most applications, Coarse Alignment to small tilt errors is readily achievable over a fairly short time period (e.g., from roll/pitch attitude inputs from another device, or by using INS accelerometers as the measure of attitude to vertical under controlled non-accelerating trajectory conditions). In some applications, Coarse Alignment to small heading error is not practical. To accommodate large initial heading uncertainties in such applications, traditional Kalman alignment structures need to be reformulated for negligible second order error impact under dynamic conditions.

This article extends the (3) quasi-stationary Fine Alignment large initial heading error approach to dynamic moving base conditions by using the [6] revised observation/measurement as a base. The article begins with the horizontal form of the [6] revised observation equation. Non-linear error equations are then developed for the observation and the (2) inertial navigation equations within a Figure 1 Fine Alignment structure. From the non-linear error equations, a Kalman filter compatible linearized error version is then developed. Lastly, the impact of linearization approximations on Fine Alignment performance is assessed by comparing the non-linear observation and linearized measurement equations forming the Figure 1 measurement residual. It is shown that by virtue of the [6] revised observation approach, all significant residual second order errors are vertical, (i.e., not present on the horizontal measurement residual), hence, have no impact on Fine Alignment performance. The article includes a summary of the equations developed to implement the large heading angle velocity matching alignment approach, including initialization of the (1) covariance matrix for Kalman gain calculations.

For simplicity and clarity in the remainder of this article, "Velocity Matching Fine Alignment" will be referred to as "Alignment" or "Kalman Alignment".

OBSERVATION EQUATION

The Observation Equation described in this article for the Kalman Alignment observation input in Figure 1 is the horizontal component of [6 - Eq. (28)]:

$$\underline{\widehat{M}}^N_n = \left(\underline{\widehat{v}}^N_n - \widehat{C}^N_{N^*_n}\, \underline{\widehat{v}}^{N^*}_{Ref_n}\right)_H \tag{4}$$

where

$\underline{\widehat{M}}^N_n$ = Observation vector in N frame navigation coordinates.

$\underline{\widehat{v}}^{N^*}_{Ref}$ = Velocity relative to the earth provided in N* frame coordinates by the reference navigation device (see Figure 1) to the Kalman filter observation block (for reference comparison with $\underline{\widehat{v}}^N$ in (2)).

$C^N_{N^*}$ = Direction cosine matrix that transforms vectors from the N* to the N frame. From the definition of N*, $C^N_{N^*}$ represents a rotation around the upward vertical.

Only the horizontal components in (4) are used in anticipation of potentially large second error propagation into the vertical observation, thereby mitigating potentially significant Kalman filter estimation errors in Figure 1. Note - For simplicity in this article, (4) does not allow for differences in physical location between the $\underline{\widehat{v}}^N$ and $\underline{\widehat{v}}^N_{Ref}$ navigation points that would normally be included in an actual system design (e.g., due to physical separation between the INS and reference navigation device under vehicle angular motion, i.e., so-called "lever arm" effects [3 - Sect. 15.2.2.2]).

Since $\widehat{C}^N_{N^*}$ is a rotation transformation around the vertical, (4) is

$$\underline{\widehat{M}}^N_n = \underline{\widehat{v}}^N_{H_n} - \widehat{C}^N_{N^*_n}\, \underline{\widehat{v}}^{N^*}_{Ref_{H/n}} \tag{5}$$

The $\underline{\widehat{v}}^N_H$ vector in (5) is calculated for this application using the following form of (2) between Kalman updates.

$$\underline{\widehat{v}}^N_{H_n} = \underline{\widehat{v}}^N_{H_{n-1}} + \int_{t_{n-1}}^{t_n} \underline{\dot{\widehat{v}}}^N_H \, dt$$
$$\underline{\dot{\widehat{v}}}^N_H = \left[\widehat{C}^N_B\, \underline{\tilde{a}}^B_{SF} + \underline{\widehat{g}}^N_P - \left(\underline{\widehat{\omega}}^N_{IN} + \underline{\widehat{\omega}}^N_{IE}\right) \times \left(\underline{\widehat{v}}^N_H + \widehat{v}_{ZN}\, \underline{u}^N_Z\right)\right]_H \tag{6}$$

with $\widehat{C}^N_B$ determined by $\dot{\widehat{C}}^N_B$ integration in (2), and where

$\hat{v}_{ZN}$ = Vertical component of $\underline{v}^N$.

Because the N and N* frame definitions make their Z vertical axes parallel, it is expedient during Alignment to set $\hat{v}_{ZN}$ in (6) to

$$\hat{v}_{ZN} = \hat{v}_{ZN^*_{Ref}} \tag{7}$$

where

$\hat{v}_{ZN^*_{Ref}}$ = Component of $\underline{\hat{v}}^{N^*}_{Ref}$ along the N* frame vertical Z axis.

Additionally, to simplify error modeling, the $\underline{\hat{g}}^N_P$ gravity vector in (4) is first calculated in N* coordinates based on reference aiding device input position data, and then transformed to the N frame:

$$\underline{\hat{g}}^N_P = \hat{C}^N_{N^*} \, \underline{\hat{g}}^{N^*}_{P_{Ref}} \tag{8}$$

where

$\underline{\hat{g}}^{N^*}_{P_{Ref}}$ = Plumb-bob gravity calculated in N* coordinates using standard INS computation techniques (e.g., [3 - Sects. 5.4 & 5.4.1]), but based on N* frame navigation data provided to the Kalman alignment process by the reference navigation device.

Similarly, for the angular rate terms in (2) and (6):

$$\begin{aligned} &\underline{\hat{\omega}}^{N^*}_{IE} = \underline{\hat{\omega}}^{N^*}_{IE_{Ref}} \qquad &&\underline{\hat{\omega}}^{N^*}_{EN} = \underline{\hat{\omega}}^{N^*}_{EN_{Ref}} \\ &\underline{\hat{\omega}}^{N}_{IE} = \hat{C}^N_{N^*} \, \underline{\hat{\omega}}^{N^*}_{IE} \qquad &&\underline{\hat{\omega}}^{N}_{EN} = \hat{C}^N_{N^*} \, \underline{\hat{\omega}}^{N^*}_{EN} \end{aligned} \tag{9}$$

where

$\underline{\hat{\omega}}^{N^*}_{IE_{Ref}}$, $\underline{\hat{\omega}}^{N^*}_{EN_{Ref}}$ = Angular rates $\underline{\omega}_{IE}$, $\underline{\omega}_{EN}$ calculated in N* coordinates using standard INS computation techniques (e.g., [3 - Sects. 4.1.1 & 5.3]), but based on N* frame navigation data provided to the Kalman alignment process by the reference navigation device.

The N relative to N* attitude in (5) represented by $C^N_{N^*}$, is a heading rotation (about the vertical) from N* to N through angle β. Thus, from [1 - Eq. (3.2.2.1-4)], $\hat{C}^N_{N^*}$ in (5) is

$$\hat{C}^N_{N^*} = I + \widehat{\sin\beta} \left(\underline{u}^N_{ZN} \times\right) + (1 - \widehat{\cos\beta}) \left(\underline{u}^N_{ZN} \times\right)\left(\underline{u}^N_{ZN} \times\right) \tag{10}$$

where

β = Constant angle measured positive around the upward defined N and N* frame Z axes.

I = Identity matrix.

Note in (10) that $\hat{C}^{N}_{N*}$ is represented by the two scalar parameters $\widehat{\sin\beta}$ and $\widehat{\cos\beta}$. The error in these parameters will form part of the errors to be estimated by the Kalman alignment filter. This is directly analogous to the [3 - Sects. 6.1.2 & 15.2.1] stationary alignment technique that uses horizontal earth rate components to represent heading, equaling horizontal earth rate magnitude multiplied by the sine and cosine of N frame heading angle from north.

Recognizing that the N frame selection maintains it at a fixed β alignment orientation relative to the N* frame, the $\hat{C}^{N}_{N*}$ matrix is constant, hence, between Figure 1 Kalman updates,

$$\frac{d}{dt}\left(\widehat{\cos\beta}\right) = 0 \qquad \frac{d}{dt}\left(\widehat{\sin\beta}\right) = 0 \tag{11}$$

It is also to be noted that a more general treatment would allow for differences between N and N* angular rates by representing C^{N}_{N*} as a heading rotation (about the vertical) from N* to N through a time changing angle α followed by the previously defined constant angle β. The α angle would be used to account for the difference in vertical rotation rates that may exist between the N and N* frames (e.g., a GPS type geographic local level east/north/up navigation frame implementation for frame N* versus a wander azimuth local level navigation N frame implementation [3 - Sect. 2.2]). However, since design of the Kalman alignment process allow selection of the N frame during alignment, it is easily set to be the same as N*. This simplifies the analytical development, allowing the N angular rate to be equated directly to the N* rate calculated from N* navigation parameter inputs (as previously shown).

OBSERVATION AS A FUNCTION OF SYSTEM ERRORS

Kalman filter theory requires that observation (5) be an unbiased error measurement so that the equivalent error free form of (5) is

$$\underline{M}^{N}_{n} = \underline{v}^{N}_{H_n} - C^{N}_{N*_n}\, \underline{v}^{N*}_{Ref_{H/n}} = 0 \tag{12}$$

Substituting for error definitions between (5) and (12) finds for (12) with (5):

$$\underline{M}_n^N = \left(\hat{\underline{v}}_{H_n}^N - \Delta\underline{v}_{H_n}^N\right) - \left(\hat{C}_{N^*n}^N - \Delta C_{N^*n}^N\right)\left(\hat{\underline{v}}_{Ref_{H/n}}^{N^*} - \delta\underline{v}_{Ref_{H/n}}^{N^*}\right) = 0$$

$$= \hat{\underline{v}}_{H_n}^N - \Delta\underline{v}_{H_n}^N - \hat{C}_{N^*n}^N \hat{\underline{v}}_{Ref_{H/n}}^{N^*} + \hat{C}_{N^*n}^N \delta\underline{v}_{Ref_{H/n}}^{N^*}$$

$$+ \Delta C_{N^*n}^N \hat{\underline{v}}_{Ref_{H/n}}^{N^*} - \Delta C_{N^*n}^N \delta\underline{v}_{Ref_{H/n}}^{N^*} \tag{13}$$

$$= \hat{\underline{M}}_n^N - \Delta\underline{v}_{H_n}^N + \hat{C}_{N^*n}^N \delta\underline{v}_{Ref_{H/n}}^{N^*} + \Delta C_{N^*n}^N \hat{\underline{v}}_{Ref_{H/n}}^{N^*} - \Delta C_{N^*n}^N \delta\underline{v}_{Ref_{H/n}}^{N^*}$$

in which the $\Delta C_{N^*}^N$, $\Delta\underline{v}_H^N$, and $\delta\underline{v}_{Ref_H}^{N^*}$ errors in $\hat{C}_{N^*}^N$, $\hat{\underline{v}}_H^N$, and $\hat{\underline{v}}_{Ref_H}^{N^*}$ are defined as

$$\Delta C_{N^*}^N \equiv \hat{C}_{N^*}^N - C_{N^*}^N \qquad \Delta\underline{v}_H^N \equiv \hat{\underline{v}}_H^N - \underline{v}_H^N \qquad \delta\underline{v}_{Ref_H}^{N^*} \equiv \hat{\underline{v}}_{Ref_H}^{N^*} - \underline{v}_{Ref_H}^{N^*} \tag{14}$$

The $\Delta(\,)$ large error assignments to $\Delta C_{N^*}^N$ and $\Delta\underline{v}_H^N$ are made because of the large initially unknown heading error between N and N*, and (as will be apparent subsequently), $\hat{\underline{v}}_H^N$ is initialized at the start of Kalman alignment using N* frame velocity data.

Then, from (13):

$$\hat{\underline{M}}_n^N = \Delta\underline{v}_{H_n}^N - \hat{C}_{N^*n}^N \delta\underline{v}_{Ref_{H/n}}^{N^*} - \Delta C_{N^*n}^N \hat{\underline{v}}_{Ref_{H/n}}^{N^*} + \Delta C_{N^*n}^N \delta\underline{v}_{Ref_{H/n}}^{N^*} \tag{15}$$

The linearized form of (15) (including linearized inputs) will form the basis for the measurement equation used in forming $\hat{\underline{z}}$ for Figure 1.

Inputs To Observation Equation (15)

The $\Delta C_{N^*}^N$ term in (15) is derived from the equivalent of (10) - (11) for the true value of $\hat{C}_{N^*}^N$:

$$C_{N^*}^N = I + \sin\beta\left(\underline{u}_{ZN}^N \times\right) + (1 - \cos\beta)\left(\underline{u}_{ZN}^N \times\right)\left(\underline{u}_{ZN}^N \times\right) \tag{16}$$

$$\frac{d}{dt}\left(\cos\beta\right) = 0 \qquad \frac{d}{dt}\left(\sin\beta\right) = 0 \tag{17}$$

From (14), the $\Delta C_{N^*}^N$ error in (15) is the difference between (10) - (11) and (16) - (17):

$$\Delta C_{N^*}^N = \hat{C}_{N^*}^N - C_{N^*}^N = \Delta\sin\beta\left(\underline{u}_{ZN}^N \times\right) - \Delta\cos\beta\left(\underline{u}_{ZN}^N \times\right)\left(\underline{u}_{ZN}^N \times\right)$$

$$\Delta\cos\beta \equiv \widehat{\cos\beta} - \cos\beta \qquad \Delta\sin\beta \equiv \widehat{\sin\beta} - \sin\beta \tag{18}$$

with, between Figure 1 Kalman updates,

$$\frac{d}{dt}\left(\Delta\cos\beta\right) = 0 \qquad \frac{d}{dt}\left(\Delta\sin\beta\right) = 0 \tag{19}$$

The $\Delta\underline{v}_H^N$ velocity error term in (15) includes Figure 1 control updates $\underline{u}_c$ and, between updates, the integral of the horizontal velocity error rate:

$$\Delta\underline{v}_{H_n}^N = \Delta\underline{v}_{H_{n-1}}^N + \int_{t_{n-1}}^{t_n} \Delta\dot{\underline{v}}_H^N \, dt \tag{20}$$

in which $\Delta\dot{\underline{v}}_H^N$ is the horizontal component of the Appendix A derived $\Delta\dot{\underline{v}}^N$ general velocity error rate:

$$\begin{aligned} \Delta\dot{\underline{v}}^N \approx \; & \hat{C}_B^N \, \delta\underline{a}_{SF}^B + \tilde{\underline{a}}_{SF}^N \times \underline{\gamma}^N + \delta\underline{g}_P^N \\ & - \left(\Delta\underline{\omega}_{IN}^N + \Delta\underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \Delta\underline{v}^N \\ & - \frac{1}{2}\left(\hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N\right) \times \underline{\gamma}^N - \left(\hat{C}_B^N \, \delta\underline{a}_{SF}^B\right) \times \underline{\gamma}^N + \left(\Delta\underline{\omega}_{IN}^N + \Delta\underline{\omega}_{IE}^N\right) \times \Delta\underline{v}^N \end{aligned} \tag{21}$$

in which

$$\begin{aligned} & \Delta\underline{\omega}_{IE}^N \equiv \hat{\underline{\omega}}_{IE}^N - \underline{\omega}_{IE}^N \qquad && \Delta\underline{\omega}_{IN}^N \equiv \hat{\underline{\omega}}_{IN}^N - \underline{\omega}_{IN}^N \\ & \delta\underline{g}_P^N \equiv \hat{\underline{g}}_P^N - \underline{g}_P^N \qquad && \delta\underline{a}_{SF}^N \equiv \tilde{\underline{a}}_{SF}^N - \underline{a}_{SF}^N \end{aligned} \tag{22}$$

and where

$\underline{\gamma}^N$ = Small angular error vector in the $\hat{C}_B^N$ attitude matrix (defined analytically in Appendix A).

The horizontal component of (21) for (20) is derived in Appendix B. The derivation follows from (7) and (9), showing that for this Kalman alignment approach, the $\Delta\underline{\omega}$ and $\Delta\underline{v}$ terms in (21) are horizontal:

$$\begin{aligned} & \Delta\underline{\omega}_{IE_H}^N = \Delta C_{N*}^N \, \hat{\underline{\omega}}_{IE_H}^{N*} \qquad \Delta\underline{\omega}_{EN_H}^N = \Delta C_{N*}^N \, \hat{\underline{\omega}}_{EN_H}^{N*} \\ & \Delta\underline{\omega}_{IN_H}^N = \Delta\underline{\omega}_{IE_H}^N + \Delta\underline{\omega}_{EN_H}^N \\ & \Delta\underline{v}^N = \Delta\underline{v}_H^N \end{aligned} \tag{23}$$

Applying (23) in (21), Appendix B then finds for horizontal velocity error rate in the (20) integrand:

$$\Delta \dot{\underline{v}}_H^N = \begin{bmatrix} \hat{C}_B^N \, \delta \underline{a}_{SF}^B + \hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N + \delta \underline{g}_P^N \\ + \underline{u}_Z^N \times \left[\hat{v}_{ZN} \left(\Delta \underline{\omega}_{IN_H}^N + \Delta \underline{\omega}_{IE_H}^N \right) - \left(\hat{\omega}_{IN_{ZN}} + \hat{\omega}_{IE_{ZN}} \right) \Delta \underline{v}_H^N \right] \\ - \frac{1}{2} \left(\hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N \right) \times \underline{\gamma}^N - \left(\hat{C}_B^N \, \delta \underline{a}_{SF}^B \right) \times \underline{\gamma}^N \end{bmatrix}_H \tag{24}$$

Note that the second order $\left(\Delta \underline{\omega}_{IN}^N + \Delta \underline{\omega}_{IE}^N \right) \times \Delta \underline{v}^N$ Coriolis term in (21) has vanished in (24) because from (23), it equals $\left(\Delta \underline{\omega}_{IN_H}^N + \Delta \underline{\omega}_{IE_H}^N \right) \times \Delta \underline{v}_H^N$ which is vertical. This is a significant finding because, due to the large nature of its errors, $\left(\Delta \underline{\omega}_{IN}^N + \Delta \underline{\omega}_{IE}^N \right) \times \Delta \underline{v}^N$ can be as large as the $\left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N \right) \times \hat{\underline{v}}^N$ term in the (2) and (6) velocity rate equations. Finding it to be vertical means that its large second order nature will not cause an error in the linearized model of (24) used in the Figure 1 Kalman filter design. Thus, it will not impact the Kalman filter measurement residual in Figure 1, and not lead to error mis-estimation. This is also a primary reason for selecting the horizontal form for observation Equation (4), i.e., to exclude the large second order vertical component of $\left(\Delta \underline{\omega}_{IN}^N + \Delta \underline{\omega}_{IE}^N \right) \times \Delta \underline{v}^N$ in the velocity error rate integral.

Based on the error in (8), the $\delta \underline{g}_P^N$ term in (24) becomes

$$\delta \underline{g}_P^N = \Delta C_{N*}^N \, \hat{\underline{g}}_{P_{Ref/H}}^{N*} \tag{25}$$

in which the minor gravity modeling error in $\underline{g}_{P_{Ref/H}}^{N*}$ has been neglected. The $\hat{\underline{g}}_{P_{Ref/H}}^{N*}$ vector in (25) would then be calculated using standard techniques (e.g., [3 - Sects. 5.4 & 5.4.1] using reference input position data. Note that $\underline{g}_{P_{Ref/H}}^{N*}$ is generally very small, on the order of one micro-g per thousand feet of altitude - The dominant one g component of gravity component is vertical. However, because of the potentially large value for ΔC_{N*}^N in this application, (25) is included in the (21) velocity error rate model (i.e., not neglected) for Kalman alignment filter design, to properly account for its presence in the Kalman filter observation equation input.

The $\underline{\gamma}^N$ attitude error term in (24) includes Figure 1 control updates $\underline{u}_c$ and, between updates, the integral of the $\dot{\underline{\gamma}}^N$ attitude error rate derived in Appendix C:

$$\dot{\underline{\gamma}}^N \approx - \hat{C}_B^N \, \delta \underline{\omega}_{IB}^B - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}^N + \Delta \underline{\omega}_{IN_H}^N + \frac{1}{2} \left(\hat{C}_B^N \, \delta \underline{\omega}_{IB}^B + \Delta \underline{\omega}_{IN_H}^N \right) \times \underline{\gamma}^N \tag{26}$$

$$\underline{\gamma}_n^N = \underline{\gamma}_{n-1}^N + \int_{t_{n-1}}^{t_n} \dot{\underline{\gamma}}^N \, dt \qquad \underline{\gamma}_0^N = \underline{\gamma}_{H_0}^N \tag{27}$$

where

0 = Subscript indicating the value of the designated parameter at the start of Kalman alignment.

Note that the initial value for $\underline{\gamma}^N$ in (27) is horizontal (i.e., zero vertical component). This is because the basic definition for the N Frame is to have a nominal initial heading, with initial heading misalignment accounted for as misalignment of the N frame relative to the N* frame.

SUMMARY OF EQUATIONS FOR KALMAN ALIGNMENT IMPLEMENTATION

The Figure 1 operations for Kalman alignment are or are derived from (1), (2), (5) - (11), (15), (18) - (20), and (23) - (27) as summarized next.

The Figure 1 INS Inertial Navigation Integration Operations For Kalman alignment between updates are (2) and (6) - (11), resequenced and renumbered next:

$$\frac{d}{dt}\left(\widehat{\cos\beta}\right) = 0 \qquad \frac{d}{dt}\left(\widehat{\sin\beta}\right) = 0 \tag{28}$$

$$\hat{C}^N_{N*} = I + \widehat{\sin\beta}\left(\underline{u}^N_{ZN}\times\right) + (1 - \widehat{\cos\beta})\left(\underline{u}^N_{ZN}\times\right)\left(\underline{u}^N_{ZN}\times\right) \tag{29}$$

$$\begin{aligned} &\hat{\underline{\omega}}^{N*}_{IE} = \hat{\underline{\omega}}^{N*}_{IE_{Ref}} \qquad &&\hat{\underline{\omega}}^{N*}_{EN} = \hat{\underline{\omega}}^{N*}_{EN_{Ref}} \\ &\hat{\underline{\omega}}^{N}_{IE} = \hat{C}^N_{N*}\,\hat{\underline{\omega}}^{N*}_{IE} \qquad &&\hat{\underline{\omega}}^{N}_{EN} = \hat{C}^N_{N*}\,\hat{\underline{\omega}}^{N*}_{EN} \end{aligned} \tag{30}$$

$$\begin{gathered} \hat{\underline{\omega}}^{N}_{IN} = \hat{\underline{\omega}}^{N}_{IE} + \hat{\underline{\omega}}^{N}_{EN} \qquad \dot{\hat{C}}^N_B = \hat{C}^N_B\left(\tilde{\underline{\omega}}^B_{IB}\times\right) - \hat{\underline{\omega}}^N_{IN}\,\hat{C}^N_B \\ \hat{C}^N_{B_n} = \hat{C}^N_{B_{n-1}} + \int_{t_{n-1}}^{t_n}\dot{\hat{C}}^N_B\,dt \end{gathered} \tag{31}$$

$$\hat{v}_{ZN} = \hat{v}_{ZN*_{Ref}} \tag{32}$$

$$\hat{\underline{g}}^N_P = \hat{C}^N_{N*}\,\hat{\underline{g}}^{N*}_{P_{Ref}} \tag{33}$$

$$\begin{gathered} \dot{\hat{\underline{v}}}^N_H = \left[\hat{C}^N_B\,\tilde{\underline{a}}^B_{SF} + \hat{\underline{g}}^N_P - \left(\hat{\underline{\omega}}^N_{IN} + \hat{\underline{\omega}}^N_{IE}\right)\times\left(\hat{\underline{v}}^N_H + \hat{v}_{ZN}\,\underline{u}^N_Z\right)\right]_H \\ \hat{\underline{v}}^N_{H_n} = \hat{\underline{v}}^N_{H_{n-1}} + \int_{t_{n-1}}^{t_n}\dot{\hat{\underline{v}}}^N_H\,dt \end{gathered} \tag{34}$$

The Observation Equation For the Kalman alignment Compare operation in Figure 1 is (5):

$$\underline{\hat{M}}^N_n = \underline{\hat{v}}^N_{H_n} - \hat{C}^N_{N*_n}\, \underline{\hat{v}}^{N*}_{Ref_{H/n}} \tag{35}$$

with

$$\hat{C}^N_{N*_0} = I \tag{36}$$

The Linearized Dynamic Error Model operations between Kalman updates in Figure 1 for Kalman alignment are the linearized forms of (18) - (20) and (23) - (27), resequenced and renumbered next:

$$\begin{gathered}\frac{d}{dt}\left(\widehat{\Delta\cos\beta}\right) = 0 \qquad \frac{d}{dt}\left(\widehat{\Delta\sin\beta}\right) = 0 \\ \widehat{\Delta C^N_{N*}} = \widehat{\Delta\sin\beta}\left(\underline{u}^N_{ZN}\times\right) - \widehat{\Delta\cos\beta}\left(\underline{u}^N_{ZN}\times\right)\left(\underline{u}^N_{ZN}\times\right)\end{gathered} \tag{37}$$

$$\begin{gathered}\widehat{\Delta\underline{\omega}^N_{IE_H}} = \widehat{\Delta C^N_{N*}}\, \underline{\hat{\omega}}^{N*}_{IE_H} \qquad \widehat{\Delta\underline{\omega}^N_{EN_H}} = \widehat{\Delta C^N_{N*}}\, \underline{\hat{\omega}}^{N*}_{EN_H} \\ \widehat{\Delta\underline{\omega}^N_{IN_H}} = \widehat{\Delta\underline{\omega}^N_{IE_H}} + \widehat{\Delta\underline{\omega}^N_{EN_H}}\end{gathered} \tag{38}$$

$$\dot{\hat{\underline{\gamma}}}^N_{Lin} = -\hat{C}^N_B\, \widehat{\delta\underline{\omega}^B_{IB}} - \underline{\hat{\omega}}^N_{IN}\times\underline{\hat{\gamma}}^N_{Lin} + \widehat{\Delta\underline{\omega}^N_{IN_H}} \tag{39}$$

$$\underline{\hat{\gamma}}^N_{Lin_n} = \underline{\hat{\gamma}}^N_{Lin_{n-1}} + \int_{t_{n-1}}^{t_n} \dot{\hat{\underline{\gamma}}}^N_{Lin}\, dt \tag{40}$$

$$\widehat{\delta\underline{g}^N_P} = \widehat{\Delta C^N_{N*}}\, \underline{\hat{g}}^{N*}_{P_{Ref/H}} \tag{41}$$

$$\widehat{\Delta\dot{\underline{v}}^N_{H_{Lin}}} = \left[\begin{gathered}\hat{C}^N_B\, \widehat{\delta\underline{a}^B_{SF}} + \underline{\hat{a}}^N_{SF}\times\widehat{\underline{\gamma}^N_{Lin}} + \widehat{\delta\underline{g}^N_P} \\ + \underline{u}^N_Z\times\left[\hat{v}_{ZN}\left(\widehat{\Delta\underline{\omega}^N_{IN_H}} + \widehat{\Delta\underline{\omega}^N_{IE_H}}\right) - \left(\hat{\omega}_{IN_{ZN}} + \hat{\omega}_{IE_{ZN}}\right)\widehat{\Delta\underline{v}^N_{H_{Lin}}}\right]\end{gathered}\right]_H \tag{42}$$

$$\widehat{\Delta\underline{v}^N_{H_{Lin/n}}} = \widehat{\Delta\underline{v}^N_{H_{Lin/n-1}}} + \int_{t_{n-1}}^{t_n} \widehat{\Delta\dot{\underline{v}}^N_{H_{Lin}}}\, dt \tag{43}$$

where

Lin = Subscript indicating linearized form of designated parameter.

The error parameters in (37), (39), (42), and for the $\delta\widehat{\underline{\omega}_{IB}^{B}}$ and $\delta\widehat{\underline{a}_{SF}^{B}}$ sensor error components, constitute the estimated error state vector $\hat{\underline{x}}$ in Figure 1. Reference [3 - Eqs. (12.5-6) & (12.5-12)] show that the estimated sensor error components can be modeled as

$$\delta\widehat{\underline{\omega}_{IB}^{B}} = \widehat{\delta K_{Scal/Mis}}\ \underline{\tilde{\omega}}_{IB}^{B} + \widehat{\delta \underline{K}_{Bias}} \qquad \delta\widehat{\underline{a}_{SF}^{B}} = \widehat{\delta L_{Scal/Mis}}\ \underline{\tilde{a}}_{SF}^{B} + \widehat{\delta \underline{L}_{Bias}} \tag{44}$$

where

> $\widehat{\delta K_{Scal/Mis}}$, $\widehat{\delta L_{Scal/Mis}}$ = Estimated gyro and accelerometer scale-factor-error/misalignment error matrices.
>
> $\widehat{\delta \underline{K}_{Bias}}$, $\widehat{\delta \underline{L}_{Bias}}$ = Estimated gyro and accelerometer bias error vectors.

The error parameter coefficients in (37), (39), (42), and (44) constitute the error state dynamic matrix $\hat{A}$ in Figure 1.

The measurement equation for the Kalman alignment Linearized Observation Model operation in Figure 1 is the linearized form of (15):

$$\hat{\underline{z}}_n^N = \widehat{\Delta \underline{v}_{H_{Lin/n}}^{N}} - \hat{C}_{N^*_n}^{N}\ \widehat{\delta \underline{v}_{Ref_{H/n}}^{N^*}} - \widehat{\Delta C_{N^*_n}^{N}}\ \hat{\underline{v}}_{Ref_{H/n}}^{N^*} \tag{45}$$

where

> $\underline{z}$ = The Measurement (see Figure 1).

The error parameter coefficients in (45) constitute the measurement matrix $\hat{H}$ in Figure 1.

The Kalman gain matrix computations for the Kalman alignment Linearized Observation Model operation in Figure 1 are from (1):

$$P_n = P_{n-1} + \int_{t_{n-1}}^{t_n} \dot{P}\, dt \qquad \dot{P} = \hat{A}\, P + P\, \hat{A}^T + \hat{G}_P\, Q_P\, \hat{G}_P^T$$

$$K_n = P_n(-)\, \hat{H}_n^T \left(\hat{H}_n\, P_n(-)\, \hat{H}_n^T + \hat{G}_{M_n}\, R_M\, \hat{G}_{M_n}^T\right)^{-1} \tag{46}$$

$$P_n(+) = \left(I - K_n\, \hat{H}_n\right) P_n(-) \left(I - K_n\, \hat{H}_n\right)^T + K_n\, \hat{G}_{M_n}\, R_M\, \hat{G}_{M_n}^T\, K_n^T$$

Note that the error parameters in (37) - (46) are shown to be estimated by the Kalman filter (indicated with a $\widehat{(\ )}$ notation), in contrast with the true ( ) values. Note also that (37) - (46) are functions of error terms and parameters containing errors, the latter available in the INS computer. This contrasts with the more typical error equations in which the navigation parameters are represented by their ideal values. Equations (37) - (46) are in the form that would actually be implemented in the INS for the Kalman filter used to estimate the error parameters during Kalman alignment. This form arises because of the general derivation process followed in this article (and in [5, 6]) by which idealized error free equations were modified to be functions of computed minus error parameters.

KALMAN ALIGNMENT INITIALIZATION

Navigation Data Initialization

The estimated attitude matrix $\hat{C}_B^N$ in (31) is initialized prior to Kalman alignment initiation at a Coarse Alignment determined value with arbitrary heading relative to north. Reference [3 - Sect. 6.1.1] illustrates a rapid Coarse Alignment method under stationary conditions. A subsequent article will describe how the equivalent fast Coarse Alignment can be performed under dynamic moving base conditions.

The estimated velocity vector $\hat{\underline{v}}_H^N$ in (34) is initialized at the horizontal velocity value provided in N* coordinates by the reference navigation device:

$$\hat{\underline{v}}_{H_0}^N = \hat{\underline{v}}_{Ref_{H/0}}^{N^*} \tag{47}$$

where it is recognized that the heading angle β between the N and N* frames is initially unknown, hence, assumed to be zero.

Estimated Error State Initialization

All error parameters in (37) - (45) are unknown at the start of Fine Alignment, hence, their estimated values within the Figure 1 estimated error state vector $\hat{\underline{x}}$ would be initialized at zero.

Covariance Matrix Initialization

The estimated covariance matrix P in Figure 1 is initialized based on the uncertainty in the $\hat{\underline{x}}$ estimated error states:

$$\underline{\chi} \equiv \hat{\underline{x}} - \underline{x} \tag{48}$$

with the covariance matrix P defined as

$$P \equiv \mathcal{E}\left(\underline{\chi}\,\underline{\chi}^T\right) \tag{49}$$

where

$\underline{\chi}$ = Uncertainty in $\hat{\underline{x}}$.

$\mathcal{E}$ = Expected value operator.

For the $\widehat{\Delta\sin\beta}$ heading error parameter in (37), the $\underline{\chi}$ component derives from β being initially unknown and equally likely to be plus or minus so that $\widehat{\sin\beta_0} = 0$. Then from the (18) definition for $\Delta\sin\beta_0$,

$$\Delta\sin\beta_0 = \widehat{\sin\beta_0} - \sin\beta_0 = - \sin\beta_0 \tag{50}$$

Since $\hat{\underline{x}}_0 = 0$ as stated previously, the initial uncertainty in $\widehat{\Delta\sin\beta}$ is zero, and the uncertainty in $\widehat{\Delta\sin\beta}$ is from (50):

$$\chi_{\Delta\sin\beta_0} = \widehat{\Delta\sin\beta_0} - \Delta\sin\beta_0 = \sin\beta_0 \tag{51}$$

Similarly, for $\widehat{\Delta\cos\beta}$,

$$\Delta\cos\beta_0 = - \cos\beta_0 \qquad \chi_{\Delta\cos\beta_0} = \cos\beta_0 \tag{52}$$

Assuming all values for β_0 have equal likelihood over the $-\pi$ to $+\pi$ range of N* to N possible headings, the corresponding (51) - (52) initial covariance elements then become

$$\begin{aligned}
\left(P_{\Delta\sin\beta\,\Delta\sin\beta}\right)_0 &= \mathcal{E}\left(\sin^2\beta_0\right) = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} \sin^2\beta_0 \; d\beta_0 \; dt = \frac{1}{2} \\
\left(P_{\Delta\cos\beta\,\Delta\cos\beta}\right)_0 &= \mathcal{E}\left(\cos^2\beta_0\right) = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} \cos^2\beta_0 \; d\beta_0 \; dt = \frac{1}{2} \\
\left(P_{\Delta\sin\beta\,\Delta\cos\beta}\right)_0 &= \left(P_{\Delta\cos\beta\,\Delta\sin\beta}\right)_0 = \mathcal{E}\left(\sin\beta_0 \cos\beta_0\right) \\
&= \frac{1}{2\,\pi}\int_{-\pi}^{\pi} \sin\beta_0 \cos\beta_0 \; d\beta_0 \; dt = 0
\end{aligned} \tag{53}$$

With velocity initialized as in (47), the initial uncertainty in $\hat{\underline{v}}^N_{H_0}$ becomes a function of the unknown β_0 between N and N*. Recognizing the true value of $\underline{v}^N_{H_0}$ to be $C^N_{N^*_0}\ \underline{v}^{N^*}_{Ref_{H/0}}$, the (14) error definition for $\Delta\underline{v}^N_{H_0}$ then finds with (47) and linearization:

$$
\begin{aligned}
\Delta\underline{v}^{N}_{H_0} &= \hat{\underline{v}}^{N}_{H_0} - \underline{v}^{N}_{H_0} = \hat{C}^{N}_{N^*_0}\,\hat{\underline{v}}^{N^*}_{Ref_{H/0}} - C^{N}_{N^*_0}\,\underline{v}^{N^*}_{Ref_{H/0}} \\
&= \left(C^{N}_{N^*_0} + \Delta C^{N}_{N^*_0}\right)\hat{\underline{v}}^{N^*}_{Ref_{H/0}} - C^{N}_{N^*_0}\left(\hat{\underline{v}}^{N^*}_{Ref_{H/0}} - \delta\underline{v}^{N^*}_{Ref_{H/0}}\right) \\
&= \Delta C^{N}_{N^*_0}\,\hat{\underline{v}}^{N^*}_{Ref_{H/0}} + C^{N}_{N^*_0}\,\delta\underline{v}^{N^*}_{Ref_{H/0}} \\
&= \Delta C^{N}_{N^*_0}\,\hat{\underline{v}}^{N^*}_{Ref_{H/0}} + \left(\hat{C}^{N}_{N^*_0} - \Delta C^{N}_{N^*_0}\right)\delta\underline{v}^{N^*}_{Ref_{H/0}} \\
&\approx \Delta C^{N}_{N^*_0}\,\hat{\underline{v}}^{N^*}_{Ref_{H/0}} + \hat{C}^{N}_{N^*_0}\,\delta\underline{v}^{N^*}_{Ref_{H/0}}
\end{aligned}
\tag{54}
$$

Because all error state estimates are initially zero, $\hat{C}^{N}_{N^*_0} = I$ and $\Delta C^{N}_{N^*_0} = \hat{C}^{N}_{N^*_0} - C^{N}_{N^*_0} = I - C^{N}_{N^*_0}$. Then (54) obtains for the linearized $\Delta\underline{v}^{N}_{H_0}$ uncertainty:

$$
\underline{\chi}_{\Delta\underline{v}H/0} = \widehat{\Delta\underline{v}^{N}_{H_0}} - \Delta\underline{v}^{N}_{H_0} = -\left(I - C^{N}_{N^*_0}\right)\hat{\underline{v}}^{N^*}_{Ref_{H/0}} - \delta\underline{v}^{N^*}_{Ref_{H/0}}
\tag{55}
$$

From (16),

$$
I - C^{N}_{N^*_0} = -\sin\beta_0\left(\underline{u}^{N}_{ZN}\times\right) - (1 - \cos\beta_0)\left(\underline{u}^{N}_{ZN}\times\right)\left(\underline{u}^{N}_{ZN}\times\right)
\tag{56}
$$

so that (55) becomes

$$
\underline{\chi}_{\Delta\underline{v}H/0} = \sin\beta_0\left(\underline{u}^{N}_{ZN}\times\hat{\underline{v}}^{N^*}_{Ref_{H/0}}\right) - (1 - \cos\beta_0)\,\hat{\underline{v}}^{N^*}_{Ref_{H/0}} - \delta\underline{v}^{N^*}_{Ref_{H/0}}
\tag{57}
$$

With (57), the horizontal velocity error initial covariance then is

$$
\begin{aligned}
P_{\Delta v H/0\,\Delta v H/0} &= \mathcal{E}\left(\underline{\chi}_{\Delta\underline{v}H/0}\,\underline{\chi}^{T}_{\Delta\underline{v}H/0}\right) \\
&= \left(\underline{u}^{N}_{ZN}\times\hat{\underline{v}}^{N^*}_{Ref_{H/0}}\right)\left(\underline{u}^{N}_{ZN}\times\hat{\underline{v}}^{N^*}_{Ref_{H/0}}\right)^{T}\mathcal{E}\left(\sin^2\beta_0\right) \\
&+ \hat{\underline{v}}^{N^*}_{Ref_{H/0}}\left(\hat{\underline{v}}^{N^*}_{Ref_{H/0}}\right)^{T}\mathcal{E}\left(1 - \cos\beta_0\right)^2 + \mathcal{E}\left[\delta\underline{v}^{N^*}_{Ref_{H/0}}\left(\delta\underline{v}^{N^*}_{Ref_{H/0}}\right)^{T}\right]
\end{aligned}
\tag{58}
$$

in which is has been assumed that β_0 and $\delta\underline{v}^{N^*}_{Ref_{H/0}}$ are uncorrelated. Over the $-\pi$ to $+\pi$ range of N* to N equally likely β_0 headings,

$$
\mathcal{E}\left(\sin^2\beta_0\right) = \frac{1}{2} \qquad \mathcal{E}\left(1 - \cos\beta_0\right)^2 = \mathcal{E}\left(1 - 2\cos\beta_0 + \cos^2\beta_0\right) = \frac{3}{2}
\tag{59}
$$

Thus, (58) becomes

$$P_{\Delta v_{H/0}\,\Delta v_{H/0}} = \frac{1}{2}\left(\underline{u}^{N}_{ZN} \times \underline{\hat{v}}^{N*}_{Ref_{H/0}}\right)\left(\underline{u}^{N}_{ZN} \times \underline{\hat{v}}^{N*}_{Ref_{H/0}}\right)^{T} + \frac{3}{2}\,\underline{\hat{v}}^{N*}_{Ref_{H/0}}\left(\underline{\hat{v}}^{N*}_{Ref_{H/0}}\right)^{T} + \mathcal{E}\left[\delta\underline{v}^{N*}_{Ref_{H/0}}\left(\delta\underline{v}^{N*}_{Ref_{H/0}}\right)^{T}\right] \qquad (60)$$

With (51) - (52) and (57), the initial cross-correlations between $\underline{\chi}_{\Delta \underline{v}_{H/0}}$, $\chi_{\Delta\sin\beta_0}$, and $\chi_{\Delta\cos\beta_0}$ are obtained similarly:

$$\begin{aligned}
\left(P_{\Delta v_H\,\Delta\sin\beta}\right)_0 &= \mathcal{E}\left(\underline{\chi}_{\Delta\underline{v}_{H/0}}\,\chi_{\Delta\sin_0}\right) = \mathcal{E}\left(\underline{\chi}_{\Delta\underline{v}_{H/0}}\,\sin\beta_0\right) \\
&= \left(\underline{u}^{N}_{ZN} \times \underline{\hat{v}}^{N*}_{Ref_{H/0}}\right)\mathcal{E}\left(\sin^2\beta_0\right) = \frac{1}{2}\,\underline{u}^{N}_{ZN} \times \underline{\hat{v}}^{N*}_{Ref_{H/0}} \\
\left(P_{\Delta\sin\beta\,\Delta v_H}\right)_0 &= \mathcal{E}\left(\chi_{\Delta\sin_0}\,\underline{\chi}^{T}_{\Delta\underline{v}_{H/0}}\right) = \mathcal{E}\left(\sin\beta_0\,\underline{\chi}^{T}_{\Delta\underline{v}_{H/0}}\right) \\
&= \left(\underline{u}^{N}_{ZN} \times \underline{\hat{v}}^{N*}_{Ref_{H/0}}\right)^{T}\mathcal{E}\left(\sin^2\beta_0\right) = \frac{1}{2}\left(\underline{u}^{N}_{ZN} \times \underline{\hat{v}}^{N*}_{Ref_{H/0}}\right)^{T}
\end{aligned} \qquad (61)$$

$$\begin{aligned}
\left(P_{\Delta v_H\,\Delta\cos\beta}\right)_0 &= \mathcal{E}\left(\underline{\chi}_{\Delta\underline{v}_{H/0}}\,\chi_{\Delta\cos_0}\right) = \mathcal{E}\left(\underline{\chi}_{\Delta\underline{v}_{H/0}}\,\cos\beta_0\right) \\
&= \underline{\hat{v}}^{N*}_{Ref_{H/0}}\,\mathcal{E}\left(\cos^2\beta_0\right) = \frac{1}{2}\,\underline{\hat{v}}^{N*}_{Ref_{H/0}} \\
\left(P_{\Delta\cos\,\Delta v_H}\right)_0 &= \mathcal{E}\left(\chi_{\Delta\cos_0}\,\underline{\chi}^{T}_{\Delta\underline{v}_{H/0}}\right) = \mathcal{E}\left(\cos\beta_0\,\underline{\chi}^{T}_{\Delta\underline{v}_{H/0}}\right) \\
&= \left(\underline{\hat{v}}^{N*}_{Ref_{H/0}}\right)^{T}\mathcal{E}\left(\cos^2\beta_0\right) = \frac{1}{2}\left(\underline{\hat{v}}^{N*}_{Ref_{H/0}}\right)^{T}
\end{aligned} \qquad (62)$$

The initial uncertainty in attitude error $\underline{\hat{\gamma}}^{N}$ recognizes that from the definition of the N frame, the vertical component of $\underline{\hat{\gamma}}^{N}$ is zero, hence, the initial uncertainty in $\underline{\hat{\gamma}}^{N}$ is horizontal, and

$$\begin{aligned}
\underline{\chi}_{\gamma_0} &= \underline{\hat{\gamma}}^{N}_{0} - \underline{\gamma}^{N}_{0} = \underline{\hat{\gamma}}^{N}_{0} - \underline{\gamma}^{N}_{H_0} = -\,\underline{\gamma}^{N}_{H_0} \\
P_{\gamma_0\,\gamma_0} &= \mathcal{E}\left(\underline{\chi}_{\gamma_0}\,\underline{\chi}^{T}_{\gamma_0}\right) = \mathcal{E}\left[\underline{\gamma}^{N}_{H_0}\left(\underline{\gamma}^{N}_{H_0}\right)^{T}\right] = I_{xy}\,\sigma^{2}_{\gamma_{H/0}}
\end{aligned} \qquad (63)$$

where

I_{xy} = Identity matrix with zero for the lower diagonal element.

$\sigma_{\gamma_{H/0}}$ = Standard deviation of the Coarse Leveling tilt uncertainty, assumed uncorrelated between horizontal axes.

The remaining sensor error covariance elements are uncorrelated between axes.

NOISE PARAMETERS

The Kalman Alignment filter linearized dynamic error models in Figure 1 derive from the general linear forms [3 - Sect. 15.1]:

$$\dot{\underline{x}} = A\ \underline{x} + G_P\ \underline{n}_P \qquad \underline{z}_n = H_n\ \underline{x}_n + G_{M_n}\ \underline{n}_{M_n} \tag{64}$$

compared to the Figure 1 estimation equivalents

$$\begin{gathered} \dot{\widehat{\underline{x}}} = \widehat{A}\ \widehat{\underline{x}} \qquad \widehat{\underline{z}}_n = \widehat{H}_n\ \widehat{\underline{x}}_n \qquad \underline{z}_{Res_n} = \underline{M}_n - \widehat{\underline{z}}_n \\ \widehat{\underline{x}}_n(+) = \widehat{\underline{x}}_n(-) + K_n\ \underline{z}_{Res_n} \end{gathered} \tag{65}$$

The uncertainty equations corresponding to (65) are derived from the (48) uncertainty definition. For uncertainty propagation between Kalman filter n cycle updates (and linearization):

$$\dot{\underline{\chi}} = \dot{\widehat{\underline{x}}} - \dot{\underline{x}} = \widehat{A}\ \widehat{\underline{x}} - A\ \underline{x} - G_P\ \underline{n}_P \approx \widehat{A}\ \widehat{\underline{x}} - \widehat{A}\ \underline{x} - \widehat{G}_P\ \underline{n}_P = \widehat{A}\ \underline{\chi} - \widehat{G}_P\ \underline{n}_P \tag{66}$$

For the error state updating operation (with linearization), at the Kalman n cycles:

$$\begin{aligned} \underline{z}_{Res_n} &= \underline{M}_n - \widehat{\underline{z}}_n \approx \underline{z}_n - \widehat{\underline{z}}_n = H_n\ \underline{x}_n + G_{M_n}\ \underline{n}_{M_n} - \widehat{H}_n\ \widehat{\underline{x}}_n(-) \\ &\approx \widehat{H}_n\ \underline{x}_n + \widehat{G}_{M_n}\ \underline{n}_{M_n} - \widehat{H}_n\ \widehat{\underline{x}}_n(-) = -\ \widehat{H}_n\ \underline{\chi}_n(-) + \widehat{G}_{M_n}\ \underline{n}_{M_n} \end{aligned} \tag{67}$$

From (65),

$$\begin{gathered} \widehat{\underline{x}}_n(+) = \widehat{\underline{x}}_n(-) + K_n\ \underline{z}_{Res_n} \\ \widehat{\underline{x}}_n(+) - \underline{x}_n = \widehat{\underline{x}}_n(-) - \underline{x}_n + K_n\ \underline{z}_{Res_n} \end{gathered} \tag{68}$$

so that with (67)

$$\begin{aligned} \underline{\chi}_n(+) &= \underline{\chi}_n(-) + K_n\ \underline{z}_{Res_n} \\ &= \underline{\chi}_n(-) + K_n \left(-\ \widehat{H}_n\ \underline{\chi}_n(-) + \widehat{G}_{M_n}\ \underline{n}_{M_n}\right) \\ &= \left(I - K_n\ \widehat{H}_n\right) \underline{\chi}_n(-) + K_n\ \widehat{G}_{M_n}\ \underline{n}_{M_n} \end{aligned} \tag{69}$$

Summarizing for the (66) and (69) uncertainty propagation and update equations:

$$\begin{gathered} \dot{\underline{\chi}} = \widehat{A}\ \underline{\chi} - \widehat{G}_P\ \underline{n}_P \\ \underline{\chi}_n(+) = \left(I - K_n\ \widehat{H}_n\right) \underline{\chi}_n(-) + K_n\ \widehat{G}_{M_n}\ \underline{n}_{M_n} \end{gathered} \tag{70}$$

By applying covariance definition (49) to (70), the equivalent $\dot{P}$ and $P_n(+)$ covariance forms of (46) are obtained. The Q_P, R_M process/measurement noise terms in (46) are, respectively, the $\underline{n}_P$ process noise density matrix, and the $\underline{n}_M$ measurement noise covariance matrix $\mathcal{E}\left(\underline{n}_M\, \underline{n}_M^T\right)$, as discussed next.

Process Noise

For the Kalman Alignment application, the Q_P process noise matrix in (46) is produced by noise components in the (23) and (33) gyro/accelerometer error vectors $\delta\underline{a}_{SF}^B$ and $\delta\underline{\omega}_{IB}^B$. From [3 - Eqs. (12.5-6) & (12.5-12)], the actual inertial sensor errors are commonly modeled as:

$$
\begin{aligned}
\delta\underline{\omega}_{IB}^B &= \delta K_{Scal/Mis}\, \underline{\omega}_{IB}^B + \delta\underline{K}_{Bias} + \delta\underline{\omega}_{Rand} \\
\delta\underline{a}_{SF}^B &= \delta L_{Scal/Mis}\, \underline{a}_{SF}^B + \delta\underline{L}_{Bias} + \delta\underline{a}_{Rand}
\end{aligned}
\tag{71}
$$

where

$\delta\underline{\omega}_{Rand}$, $\delta\underline{a}_{Rand}$ = Gyro and accelerometer random noise error components.

The velocity and attitude error state uncertainties are

$$
\underline{\chi}_{\Delta v_H} = \Delta\hat{\underline{v}}_H^N - \Delta\underline{v}_H^N \qquad \underline{\chi}_\gamma = \hat{\underline{\gamma}}^N - \underline{\gamma}^N
\tag{72}
$$

so that

$$
\dot{\underline{\chi}}_{\Delta v_H} = \Delta\dot{\hat{\underline{v}}}_H^N - \Delta\dot{\underline{v}}_H^N \qquad \dot{\underline{\chi}}_\gamma = \dot{\hat{\underline{\gamma}}}^N - \dot{\underline{\gamma}}^N
\tag{73}
$$

Substituting (39) and (42) for $\dot{\hat{\underline{\gamma}}}^N$, $\Delta\dot{\hat{\underline{v}}}_H^N$ and their equivalent true value form from the linearized versions of (26) and (24), then finds from (73) with (44) and (71):

$$
\begin{aligned}
\dot{\underline{\chi}}_\gamma &= \dot{\hat{\underline{\gamma}}}^N - \dot{\underline{\gamma}}^N = \hat{\underline{A}}_{\gamma(woSens)}\, \underline{\chi} + \hat{C}_B^N \left(\delta\hat{\underline{\omega}}_{IB}^B - \delta\underline{\omega}_{IB}^B\right) \\
&= \hat{\underline{A}}_\gamma\, \underline{\chi} + \hat{C}_B^N\, \delta\underline{\omega}_{Rand} \\
\dot{\underline{\chi}}_{\Delta v_H} &= \Delta\dot{\hat{\underline{v}}}_H^N - \Delta\dot{\underline{v}}_H^N = \hat{\underline{A}}_{\Delta v_H(woSens)}\, \underline{\chi} - \hat{C}_B^N \left(\delta\hat{\underline{a}}_{SF}^B - \delta\underline{a}_{SF}^B\right) \\
&= \hat{\underline{A}}_{\Delta v_H}\, \underline{\chi} - \hat{C}_B^N\, \delta\underline{a}_{Rand}
\end{aligned}
\tag{74}
$$

For $\delta\underline{\omega}_{Rand}$, $\delta\underline{a}_{Rand}$ typically modeled as white noise, the associated (46) elements in Q_P would be along the diagonal and equal to the corresponding white noise densities [3 - Sect. 15.1.2.1.1]. From (74), the corresponding $\hat{G}_P$ process noise coupling matrix would be $\hat{C}_B^N$. Note, however, that because the transpose of a correctly computed direction cosine matrix equals its transpose, and because Q_P is diagonal (uncorrelated process noise elements in $\delta\underline{\omega}_{Rand}$, $\delta\underline{a}_{Rand}$), if the noise vector components have equal densities, the $\hat{G}_P \; Q_P \; \hat{G}_P^T$ term in (46) would reduce to $\hat{G}_P$.

It is also to be noted that sensor quantization noise present on the sensor outputs is treated differently, predominantly modeled as random uncertainty in attitude (for gyro quantization) and velocity (for accelerometer quantization) [3 - Sect. 12.5]. As such, gyro quantization becomes predominantly a white noise input to the velocity error uncertainty rate equation, modeled as a white noise density in the Q_P process noise matrix [3 - Eq. (12.5-18)]. For accelerometer error, the comparable treatment models quantization as white noise input to the position error uncertainty rate equation, not implemented for the velocity matching alignment approach described herein. However, accelerometer quantization noise does also become measurement noise for a velocity type measurement (described in the next section). For the integrated velocity matching alignment approach described in [3 - Sect. 15.2.2], accelerometer quantization would be modeled as white process noise input to the integrated velocity matching measurement integrator.

Measurement Noise

For the Alignment Kalman filter gain design, the measurement residual in Figure 1 is from (45), (15), and [3 - Eq. (12.5-11)], with linearization

$$
\begin{aligned}
\underline{z}_{Res_n} = \hat{\underline{M}}_n^N - \hat{\underline{z}}_n^N \approx \underline{z}_n^N - \hat{\underline{z}}_n^N &= - \hat{H}_{wo\Delta v_{Hn}} \, \underline{\chi}_{wo\Delta v_{H_n}}(-) - \left(\widehat{\Delta \underline{v}_{H_n}^N}(-) - \Delta \underline{v}_{H_n}^N \right) \\
&= - \hat{H}_n \, \underline{\chi}_n(-) + \hat{C}_{B_n}^N \, \delta\underline{\upsilon}_{Quant}
\end{aligned}
\tag{75}
$$

where

$\underline{z}_n^N$ = Linearized form of $\hat{\underline{M}}_n^N$ including linearized $\hat{\underline{M}}_n^N$ components.

Then, as in (69),

$$
\begin{aligned}
\underline{\chi}_n(+) = \underline{\chi}_n(-) + K_n \, \underline{z}_{Res_n} &= \underline{\chi}_n(-) + K_n \left(- \hat{H}_n \, \underline{\chi}_n(-) + \hat{C}_{B_n}^N \, \delta\underline{\upsilon}_{Quant} \right) \\
&= \left(I - K_n \, \hat{H}_n \right) \underline{\chi}_n(-) + K_n \, \hat{C}_{B_n}^N \, \delta\underline{\upsilon}_{Quant}
\end{aligned}
\tag{76}
$$

where

$\delta\underline{\upsilon}_{Quant}$ = accelerometer quantization noise.

For $\delta\underline{\upsilon}_{Quant}$ in (76) typically modeled as a white vector sequence, the (76) covariance equivalent in (46) sets the elements of measurement noise matrix R_M equal to the

$\underline{\delta\upsilon}_{Quant}$ covariance: $\mathcal{E}\left(\underline{\delta\upsilon}_{Quant}\ \underline{\delta\upsilon}_{Quant}^{T}\right)$. From (76), the corresponding $\hat{G}_{M_n}$ measurement noise coupling matrix in (46) would be $\hat{C}_B^N$. Because the transpose of a correctly computed direction cosine matrix equals its transpose, and because R_M is diagonal (uncorrelated measurement noise elements in $\underline{\delta\upsilon}_{Quant}$), if the noise vector components have equal variances, the $\hat{G}_{M_n}\,R_M\,\hat{G}_{M_n}^T$ term in (46) would reduce to R_M.

NAVIGATION MODE ENTRY AT ALIGNMENT COMPLETION

Kalman alignment completion is defined as the time when $\Delta\sin\beta$ is reduced to an acceptable value (as manifested in the $P_{\Delta\sin\beta\,\Delta\sin\beta}$ covariance element of P). This minimizes the error in $\hat{C}_{N*}^N$ which then will accurately represent the heading orientation of the N frame relative to the N* frame. Initialization of attitude for inertial navigation would proceed based on the values for $\hat{C}_{N*}^N$ and $\hat{C}_B^N$ at Kalman alignment completion. For example, attitude for inertial navigation could be initialized by resetting the $\hat{C}_B^N$ matrix to have its Y axis parallel to the N* frame Y axis using:

$$\hat{C}_B^N(+) = \left(\hat{C}_{N*}^N\right)_{AlnEnd}^T\,\hat{C}_B^N(-) \tag{77}$$

where

$\hat{C}_B^N(-) = \hat{C}_B^N$ at Kalman Alignment completion.

$\left(\hat{C}_{N*}^N\right)_{AlnEnd} = \hat{C}_{N*}^N$ at Kalman Alignment completion.

$\hat{C}_B^N(+) = \hat{C}_B^N$ to initiate inertial navigation after rotating $\hat{C}_B^N(-)$ about the vertical so that the N frame Y axis aligns with the N* frame Y axis.

Following navigation mode initialization at the end of alignment, the $\hat{C}_B^N$ matrix would be updated as in (2) using normal methods typically employed (e.g., with the N frame defined as the azimuth wander type whereby the vertical component of $\hat{\underline{\omega}}_{IN}^N$ would be set to the vertical component of earth rate, as in [3 - Sect. 4.5]).

SECOND ORDER ERROR IMPACT ASSESSMENT

The impact of second order errors on the Kalman alignment process can be assessed by analyzing the linear compared with the non-linear terms in the (15) observation equation. To simplify the analysis, we will analyze (15) when the $\hat{\underline{u}}_c$ control vector is zero. This converts the Kalman filter in Figure 1 to a pure estimator in which the errors in the observation propagate in the normal manner without Kalman intervention. To further simplify the analysis, sensor, reference input, and gravity error terms will be neglected to obtain from (15), (19), (20), (24), (26), and (27) for the relevant equations:

$$\dot{\underline{\gamma}}^N \approx - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}^N + \Delta\underline{\omega}_{IN_H}^N + \frac{1}{2}\,\Delta\underline{\omega}_{IN_H}^N \times \underline{\gamma}^N \tag{78}$$

$$\underline{\gamma}^N = \underline{\gamma}_{H_0}^N + \int_0^t \dot{\underline{\gamma}}^N \, d\tau \tag{79}$$

$$\Delta\dot{\underline{v}}_H^N \approx \begin{bmatrix} \hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N \\ + \underline{u}_Z^N \times \left[\hat{v}_{ZN}\left(\Delta\underline{\omega}_{IN_H}^N + \Delta\underline{\omega}_{IE_H}^N\right) - \left(\hat{\omega}_{IN_{ZN}} + \hat{\omega}_{IE_{ZN}}\right)\Delta\underline{v}_H^N\right] \\ - \frac{1}{2}\left(\hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N\right) \times \underline{\gamma}^N \end{bmatrix}_H \tag{80}$$

$$\Delta\underline{v}_{H_n}^N = \Delta\underline{v}_{H_0}^N + \int_0^{t_n} \Delta\dot{\underline{v}}_H^N \, dt \tag{81}$$

$$\frac{d}{dt}\left(\Delta\cos\beta\right) = 0 \qquad \frac{d}{dt}\left(\Delta\sin\beta\right) = 0 \tag{82}$$

$$\hat{\underline{M}}_n^N \approx \Delta\underline{v}_{H_n}^N - \Delta C_{N*_0}^N \, \hat{\underline{v}}_{Ref_{H/n}}^{N*} \tag{83}$$

Note in (83) that the initial value of ΔC_{N*}^N is used based on (82) and the assumption of analysis without zero error controls in Figure 1.

To analytically solve the equations, a first order Picard expansion approach is employed in which the $\underline{\gamma}^N$ term in (78) is approximated by its initial value. Then (79) integrates to

$$\begin{aligned} \underline{\gamma}^N = \underline{\gamma}_{H_0}^N + \int_0^t \dot{\underline{\gamma}}^N \, d\tau \approx \underline{\gamma}_{H_0}^N + \int_0^t \Delta\underline{\omega}_{IN_H}^N \, d\tau + \underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN}^N \, d\tau \\ - \frac{1}{2}\,\underline{\gamma}_{H_0}^N \times \int_0^t \Delta\underline{\omega}_{IN_H}^N \, d\tau \end{aligned} \tag{84}$$

Using the first order Picard expansion approach in (80) finds with (84):

$$
\Delta \dot{\underline{v}}_H^N \approx \left[\begin{array}{c} \hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N \\ + \underline{u}_Z^N \times \left[\hat{v}_{ZN} \left(\Delta \underline{\omega}_{IN_H}^N + \Delta \underline{\omega}_{IE_H}^N \right) - \left(\hat{\omega}_{ZN} + \hat{\omega}_{IE} \right) \Delta \underline{v}_{H_0}^N \right] \\ - \frac{1}{2} \left(\hat{\underline{a}}_{SF}^N \times \underline{\gamma}_{H_0}^N \right) \times \underline{\gamma}_{H_0}^N \end{array} \right]_H
$$

$$
= \left[\begin{array}{c} \hat{\underline{a}}_{SF}^N \times \left(\begin{array}{c} \underline{\gamma}_{H_0}^N + \int_0^t \Delta \underline{\omega}_{IN_H}^N \, d\tau + \underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN}^N \, d\tau \\ - \frac{1}{2} \underline{\gamma}_{H_0}^N \times \int_0^t \Delta \underline{\omega}_{IN_H}^N \, d\tau \end{array} \right) \\ + \underline{u}_Z^N \times \left[\hat{v}_{ZN} \left(\Delta \underline{\omega}_{IN_H}^N + \Delta \underline{\omega}_{IE_H}^N \right) - \left(\hat{\omega}_{IN_{ZN}} + \hat{\omega}_{IE_{ZN}} \right) \Delta \underline{v}_{H_0}^N \right] \\ - \frac{1}{2} \left(\hat{\underline{a}}_{SF}^N \times \underline{\gamma}_{H_0}^N \right) \times \underline{\gamma}_{H_0}^N \end{array} \right]_H \qquad (85)
$$

$$
\begin{aligned}
&= \hat{a}_{SF_{ZN}} \underline{u}_Z^N \times \left(\underline{\gamma}_{H_0}^N + \int_0^t \Delta \underline{\omega}_{IN_H}^N \, d\tau \right) + \left[\hat{\underline{a}}_{SF}^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN}^N \, d\tau \right) \right]_H \\
&\quad + \underline{u}_Z^N \times \left[\hat{v}_{ZN} \left(\Delta \underline{\omega}_{IN_H}^N + \Delta \underline{\omega}_{IE_H}^N \right) - \left(\hat{\omega}_{IN_{ZN}} + \hat{\omega}_{IE_{ZN}} \right) \Delta \underline{v}_{H_0}^N \right] \\
&\quad - \frac{1}{2} \left[\hat{\underline{a}}_{SF}^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \Delta \underline{\omega}_{IN_H}^N \, d\tau \right) \right]_H - \frac{1}{2} \left(\hat{\underline{a}}_{SF_H}^N \times \underline{\gamma}_{H_0}^N \right) \times \underline{\gamma}_{H_0}^N
\end{aligned}
$$

The $\hat{\underline{a}}_{SF}^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN}^N \, dt \right)$ term in (85) expands as

$$
\begin{aligned}
&\hat{\underline{a}}_{SF}^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN}^N \, d\tau \right) \\
&= \left(\hat{\underline{a}}_{SF_H}^N + \hat{a}_{SF_{ZN}} \underline{u}_Z^N \right) \times \left\{ \underline{\gamma}_{H_0}^N \times \left[\int_0^t \hat{\underline{\omega}}_{IN_H}^N \, d\tau + \left(\int_0^t \hat{\omega}_{IN_{ZN}} \, d\tau \right) \underline{u}_Z^N \right] \right\} \\
&= \hat{\underline{a}}_{SF_H}^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN_H}^N \, d\tau \right) + \hat{\underline{a}}_{SF_H}^N \times \left[\underline{\gamma}_{H_0}^N \times \left(\int_0^t \hat{\omega}_{IN_{ZN}} \, d\tau \right) \underline{u}_Z^N \right] \\
&\quad + \hat{a}_{SF_{ZN}} \underline{u}_Z^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN_H}^N \, d\tau \right) + \hat{a}_{SF_{ZN}} \underline{u}_Z^N \times \left[\underline{\gamma}_{H_0}^N \times \left(\int_0^t \hat{\omega}_{IN_{ZN}} \, d\tau \right) \underline{u}_Z^N \right] \\
&= \hat{\underline{a}}_{SF_H}^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN_H}^N \, d\tau \right) + \hat{\underline{a}}_{SF_H}^N \times \left[\underline{\gamma}_{H_0}^N \times \left(\int_0^t \hat{\omega}_{IN_{ZN}} \, d\tau \right) \underline{u}_Z^N \right] \\
&\quad + \hat{a}_{SF_{ZN}} \underline{u}_Z^N \times \left[\underline{\gamma}_{H_0}^N \times \left(\int_0^t \hat{\omega}_{IN_{ZN}} \, d\tau \right) \underline{u}_Z^N \right] \\
&= \hat{\underline{a}}_{SF_H}^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN_H}^N \, d\tau \right) + \hat{\underline{a}}_{SF_H}^N \times \left[\underline{\gamma}_{H_0}^N \times \left(\int_0^t \hat{\omega}_{IN_{ZN}} \, d\tau \right) \underline{u}_Z^N \right] \\
&\quad - \hat{a}_{SF_{ZN}} \left(\int_0^t \hat{\omega}_{IN_{ZN}} \, d\tau \right) \underline{\gamma}_{H_0}^N
\end{aligned} \qquad (86)
$$

so that

$$
\begin{aligned}
&\left[\hat{\underline{a}}_{SF}^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN}^N \, d\tau \right) \right]_H \\
&= \hat{\underline{a}}_{SF_H}^N \times \left(\underline{\gamma}_{H_0}^N \times \int_0^t \hat{\underline{\omega}}_{IN_H}^N \, d\tau \right) - \hat{a}_{SF_{ZN}} \left(\int_0^t \hat{\omega}_{IN_{ZN}} \, d\tau \right) \underline{\gamma}_{H_0}^N
\end{aligned} \qquad (87)
$$

For the $\frac{1}{2}\left[\underline{\hat{a}}_{SF}^{N} \times \left(\underline{\gamma}_{H_0}^{N} \times \int_0^t \Delta\underline{\omega}_{IN_H}^{N}\, dt\right)\right]_H$ term in (85), using (23) for $\Delta\underline{\omega}_{IN_H}^{N}$ and $\Delta C_{N*}^{N} = \Delta C_{N*_0}^{N}$ due to no-control operation:

$$\frac{1}{2}\left[\underline{\hat{a}}_{SF}^{N} \times \left(\underline{\gamma}_{H_0}^{N} \times \int_0^t \Delta\underline{\omega}_{IN_H}^{N}\, dt\right)\right]_H = \frac{1}{2}\, \underline{\hat{a}}_{SF_H}^{N} \times \left[\underline{\gamma}_{H_0}^{N} \times \left(\Delta C_{N*_0}^{N} \int_0^t \underline{\hat{\omega}}_{IN_H}^{N*}\, d\tau\right)\right] \tag{88}$$

Then using (87) - (88) in (85), with (23) for $\Delta\underline{\omega}_{IN_H}^{N}$, $\Delta\underline{\omega}_{IE_H}^{N}$ and $\Delta C_{N*}^{N} = \Delta C_{N*_0}^{N}$ due to no-control operation, the (81) integration yields

$$\begin{aligned}
\Delta\underline{v}_{H_n}^{N} &= \Delta\underline{v}_{H_0}^{N} + \int_0^{t_n} \Delta\dot{\underline{v}}_{H}^{N}\, dt \\
&= \Delta\underline{v}_{H_0}^{N} + \left(\int_0^{t_n} \hat{a}_{SF_{ZN}}\, dt\right) \underline{u}_Z^{N} \times \underline{\gamma}_{H_0}^{N} \\
&\quad + \Delta C_{N*_0}^{N} \int_0^{t_n} \left(\hat{a}_{SF_{ZN}}\, \underline{u}_Z^{N} \times \int_0^t \Delta\underline{\omega}_{IN_H}^{N*}\, d\tau\right) dt \\
&\quad + \int_0^{t_n} \left[\underline{\hat{a}}_{SF_H}^{N} \times \left(\underline{\gamma}_{H_0}^{N} \times \int_0^t \underline{\hat{\omega}}_{IN_H}^{N}\, d\tau\right)\right] dt - \int_0^{t_n} \left[\hat{a}_{SF_{ZN}} \left(\int_0^t \hat{\omega}_{IN_{ZN}}\, d\tau\right) dt\right] \underline{\gamma}_{H_0}^{N} \\
&\quad + \underline{u}_Z^{N} \times \int_0^{t_n} \left[\hat{v}_{ZN}\, \Delta C_{N*_0}^{N} \left(\Delta\underline{\omega}_{IN_H}^{N*} + \Delta\underline{\omega}_{IE_H}^{N*}\right) - \left(\hat{\omega}_{IN_{ZN}} + \hat{\omega}_{IE_{ZN}}\right) \Delta\underline{v}_{H_0}^{N}\right] dt \\
&\quad - \frac{1}{2} \int_0^{t_n} \left\{\underline{\hat{a}}_{SF_H}^{N} \times \left[\underline{\gamma}_{H_0}^{N} \times \left(\Delta C_{N*_0}^{N} \int_0^t \underline{\hat{\omega}}_{IN_H}^{N*}\, d\tau\right)\right]\right\} dt \\
&\quad + \frac{1}{2} \left(\underline{\gamma}_{H_0}^{N} \times \int_0^{t_n} \underline{\hat{a}}_{SF_H}^{N}\, dt\right) \times \underline{\gamma}_{H_0}^{N}
\end{aligned} \tag{89}$$

Since the heading orientation between the N and N* frames is unknown at the start of Kalman alignment, $\hat{C}_{N*_0}^{N} = I$, and without Figure 1 control updates, $\hat{C}_{N*}^{N} = \hat{C}_{N*_0}^{N} = I$. Thus, $\int_0^{t_n} \underline{\hat{a}}_{SF_H}^{N} = \underline{\hat{v}}_{H_n}^{N} - \underline{\hat{v}}_{H_0}^{N} = \hat{C}_{N*_0}^{N}\left(\underline{\hat{v}}_{Ref_{H/n}}^{N*} - \underline{\hat{v}}_{Ref_{H/0}}^{N*}\right) = \underline{\hat{v}}_{Ref_{H/n}}^{N*} - \underline{\hat{v}}_{Ref_{H/0}}^{N*}$ and, with (54) (neglecting reference velocity errors), (89) becomes

$$\begin{aligned}
\Delta\underline{v}_{H_n}^{N} &= \Delta C_{N*_0}^{N}\, \underline{v}_{Ref_{H/0}}^{N*} + \left(\int_0^{t_n} \hat{a}_{SF_{ZN}}\, dt\right) \underline{u}_Z^{N} \times \underline{\gamma}_{H_0}^{N} \\
&\quad + \Delta C_{N*_0}^{N} \int_0^{t_n} \left(\hat{a}_{SF_{ZN}}\, \underline{u}_Z^{N} \times \int_0^t \Delta\underline{\omega}_{IN_H}^{N*}\, d\tau\right) dt \\
&\quad + \int_0^{t_n} \left[\underline{\hat{a}}_{SF_H}^{N} \times \left(\underline{\gamma}_{H_0}^{N} \times \int_0^t \underline{\hat{\omega}}_{IN_H}^{N}\, d\tau\right)\right] dt - \int_0^{t_n} \left[\hat{a}_{SF_{ZN}} \left(\int_0^t \hat{\omega}_{IN_{ZN}}\, d\tau\right) dt\right] \underline{\gamma}_{H_0}^{N} \\
&\quad + \underline{u}_Z^{N} \times \int_0^{t_n} \left[\hat{v}_{ZN}\, \Delta C_{N*_0}^{N} \left(\Delta\underline{\omega}_{IN_H}^{N*} + \Delta\underline{\omega}_{IE_H}^{N*}\right) - \left(\hat{\omega}_{IN_{ZN}} + \hat{\omega}_{IE_{ZN}}\right) \Delta\underline{v}_{H_0}^{N}\right] dt \\
&\quad - \frac{1}{2} \int_0^{t_n} \left\{\underline{\hat{a}}_{SF_H}^{N} \times \left[\underline{\gamma}_{H_0}^{N} \times \left(\Delta C_{N*_0}^{N} \int_0^t \underline{\hat{\omega}}_{IN_H}^{N*}\, d\tau\right)\right]\right\} dt \\
&\quad + \frac{1}{2} \left[\underline{\gamma}_{H_0}^{N} \times \left(\underline{v}_{Ref_{H/n}}^{N*} - \underline{v}_{Ref_{H/0}}^{N*}\right)\right] \times \underline{\gamma}_{H_0}^{N}
\end{aligned} \tag{90}$$

Finally, (90) is substituted into (83) to obtain for the observation:

$$
\begin{aligned}
\hat{\underline{M}}^N_n &\approx \Delta\underline{v}^N_{H_n} - \Delta C^N_{N*_0}\, \hat{\underline{v}}^{N*}_{Ref_{H/n}} \\
&= - \Delta C^N_{N*_0} \left(\hat{\underline{v}}^{N*}_{Ref_{H/n}} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right) + \left(\int_0^{t_n} \hat{a}_{SF_{ZN}}\, dt \right) \underline{u}^N_Z \times \underline{\gamma}^N_{H_0} \\
&\quad + \Delta C^N_{N*_0} \int_0^{t_n} \left(\hat{a}_{SF_{ZN}}\, \underline{u}^N_Z \times \int_0^t \underline{\omega}^{N*}_{IN_H}\, d\tau \right) dt \\
&\quad + \int_0^{t_n} \left[\hat{\underline{a}}^N_{SF_H} \times \left(\underline{\gamma}^N_{H_0} \times \int_0^t \hat{\underline{\omega}}^N_{IN_H}\, d\tau \right) \right] dt - \left[\int_0^{t_n} \hat{a}_{SF_{ZN}} \left(\int_0^t \hat{\omega}_{IN_{ZN}}\, d\tau \right) dt \right] \underline{\gamma}^N_{H_0} \qquad (91) \\
&\quad + \underline{u}^N_Z \times \int_0^{t_n} \left[\hat{v}_{ZN}\, \Delta C^N_{N*_0} \left(\underline{\omega}^{N*}_{IN_H} + \underline{\omega}^{N*}_{IE_H} \right) - \left(\hat{\omega}^{N*}_{IN_{ZN}} + \hat{\omega}^{N*}_{IE_{ZN}} \right) \Delta\underline{v}^N_{H_0} \right] dt \\
&\quad - \frac{1}{2} \int_0^{t_n} \left\{ \hat{\underline{a}}^N_{SF_H} \times \left[\underline{\gamma}^N_{H_0} \times \left(\Delta C^N_{N*_0} \int_0^t \hat{\underline{\omega}}^N_{IN_H}\, d\tau \right) \right] \right\} dt \\
&\quad + \frac{1}{2} \left[\underline{\gamma}^N_{H_0} \times \left(\hat{\underline{v}}^{N*}_{Ref_{H/n}} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right) \right] \times \underline{\gamma}^N_{H_0}
\end{aligned}
$$

The last two terms in (91) constitute the non-linearities that are not accounted for in the linearized Kalman filter equations. Their impact on performance can be assessed by comparison with (91) linear terms having similar signatures.

The last term in (91) is on the order of $\gamma^2_{H_0} \left(\hat{\underline{v}}^{N*}_{Ref_{H/n}} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right)$ in magnitude. The comparable linear term in (91) is $\Delta C^N_{N*_0} \left(\hat{\underline{v}}^{N*}_{Ref_{H/n}} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right)$. Clearly, $\gamma^2_{H_0} \left(\hat{\underline{v}}^{N*}_{Ref_{H/n}} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right)$ is second order compared with $\Delta C^N_{N*_0} \left(\hat{\underline{v}}^{N*}_{Ref_{H/n}} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right)$, hence, negligible in comparison. Similarly, compared with the linear $\Delta C^N_{N*_0} \left(\hat{\underline{v}}^{N*}_{Ref_{H/n}} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right)$ term, the second to last term in (91) is on the order of $\frac{1}{2} \gamma_{H_0} \left(\int_0^t \hat{\omega}^N_{IN_H}\, d\tau \right) \Delta C^N_{N*_0} \left(\hat{\underline{v}}^{N*}_{Ref_{H/n}} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right)$ in magnitude, hence, also negligibly small (particularly since $\int_0^t \hat{\omega}^N_{IN_H}\, d\tau$ is also small during Kalman alignment). Thus, the impact of second order terms on the input observation should have negligible impact on linearized Kalman alignment performance.

APPENDIX A - VELOCITY ERROR RATE EQUATION DERIVATION

The idealized error free form of the velocity rate in (2) is

$$
\dot{\underline{v}}^N = C^N_B\, \underline{a}^B_{SF} + \underline{g}^N_P - \left(\underline{\omega}^N_{IN} + \underline{\omega}^N_{IE} \right) \times \underline{v}^N \qquad \text{(A-1)}
$$

The errors between the computed equivalents in (2) and (A-1) are defined as

$$\delta \underline{C}_B^N \equiv \hat{C}_B^N - C_B^N \qquad \Delta\underline{\omega}_{IE}^N \equiv \hat{\underline{\omega}}_{IE}^N - \underline{\omega}_{IE}^N \qquad \Delta\underline{\omega}_{IN}^N \equiv \hat{\underline{\omega}}_{IN}^N - \underline{\omega}_{IN}^N$$
$$\delta\underline{g}_P^N \equiv \hat{\underline{g}}_P^N - \underline{g}_P^N \qquad \delta\underline{a}_{SF}^N \equiv \tilde{\underline{a}}_{SF}^N - \underline{a}_{SF}^N \tag{A-2}$$

in which the errors in $\hat{\underline{\omega}}_{IE}^N$ and $\hat{\underline{\omega}}_{IN}^N$ are anticipated to be comparable in magnitude to $\underline{\omega}_{IE}^N$ and $\underline{\omega}_{IN}^N$ due to large heading uncertainties in their (9) computation. In contrast, the error in $\hat{\underline{g}}_P^N$ is small relative to $\underline{g}_P^N$ because it is primarily along the local vertical as is the equivalent in N* frame coordinates, hence is largely unaffected by the $\hat{C}_{N*}^N$ transformation in its (8) computation. Based on the (A-2) definitions, (A-1) can be expanded as:

$$\begin{aligned}
&\hat{\dot{\underline{v}}}^N - \Delta\dot{\underline{v}}^N = \left(\hat{C}_B^N - \delta C_B^N\right)\left(\tilde{\underline{a}}_{SF}^B - \delta\underline{a}_{SF}^B\right) \\
&+ \hat{\underline{g}}_P^N - \delta\underline{g}_P^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N - \Delta\underline{\omega}_{IN}^N - \Delta\underline{\omega}_{IE}^N\right) \times \left(\hat{\underline{v}}^N - \Delta\underline{v}^N\right) \\
&= \hat{C}_B^N \tilde{\underline{a}}_{SF}^B - \delta C_B^N \tilde{\underline{a}}_{SF}^B - \left(\hat{C}_B^N - \delta C_B^N\right)\delta\underline{a}_{SF}^B + \hat{\underline{g}}_P^N - \delta\underline{g}_P^N \\
&\quad - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \hat{\underline{v}}^N + \left(\Delta\underline{\omega}_{IN}^N + \Delta\underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N \\
&\quad + \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \Delta\underline{v}^N - \left(\Delta\underline{\omega}_{IN}^N + \Delta\underline{\omega}_{IE}^N\right) \times \Delta\underline{v}^N
\end{aligned} \tag{A-3}$$

or with $\hat{\dot{\underline{v}}}^N$ from (2)

$$\begin{aligned}
&\Delta\dot{\underline{v}}^N = \delta C_B^N \tilde{\underline{a}}_{SF}^B + \hat{C}_B^N \delta\underline{a}_{SF}^B + \delta\underline{g}_P^N \\
&- \left(\Delta\underline{\omega}_{IN}^N + \Delta\underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \Delta\underline{v}^N \\
&- \delta C_B^N \delta\underline{a}_{SF}^B + \left(\Delta\underline{\omega}_{IN}^N + \Delta\underline{\omega}_{IE}^N\right) \times \Delta\underline{v}^N
\end{aligned} \tag{A-4}$$

The δC_B^N error in (A-4) is derived from

$$\delta C_B^N \equiv \hat{C}_B^N - C_B^N = \left[I - C_B^N \left(\hat{C}_B^N\right)^T\right] \hat{C}_B^N \tag{A-5}$$

Applying [3 - Eq. (3.5.2-8)] assigns the cause for the δC_B^N error to misalignment of the N frame from its nominal error free value. Identifying the misaligned N frame as $\hat{N}$ gives:

$$\hat{C}_B^N = C_B^{\hat{N}} = C_N^{\hat{N}} C_B^N \tag{A-6}$$

or

$$C_B^N = C_{\hat{N}}^N \hat{C}_B^N \qquad \text{(A-7)}$$

hence, with (A-7) in (A-5),

$$\delta C_B^N = \left(I - C_{\hat{N}}^N\right) \hat{C}_B^N \qquad \text{(A-8)}$$

Defining $C_{\hat{N}}^N$ in terms of a rotation vector using the form of [3 - Eq. (19.1.3-3)] as a model:

$$C_{\hat{N}}^N = I + f_1(\gamma) \left(\underline{\gamma}^N \times\right) + f_2(\gamma) \left(\underline{\gamma}^N \times\right) \left(\underline{\gamma}^N \times\right)$$
$$f_1(\gamma) = \frac{\sin \gamma}{\gamma} = 1 - \frac{\gamma^2}{3\,!} + \cdots \qquad f_2(\gamma) = \frac{\left(1 - \cos \gamma\right)}{\gamma^2} = \frac{1}{2} - \frac{\gamma^2}{4\,!} + \cdots \qquad \text{(A-9)}$$

where

$\underline{\gamma}^N$ = Rotation angle error vector associated with the $\hat{C}_B^N$ matrix considering the N frame to be misaligned, as projected on frame N axes.

Applying (A-9) in (A-8) yields with no approximations:

$$\delta C_B^N = - \left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right) \left(\underline{\gamma}^N \times\right)\right] \hat{C}_B^N \qquad \text{(A-10)}$$

Using (A-10) for δC_B^N in (A-4) finds with no approximations:

$$\begin{aligned}
\Delta \dot{\underline{v}}^N = {} & \hat{C}_B^N \, \delta \underline{a}_{SF}^B + f_1 \left(\hat{C}_B^N \, \tilde{\underline{a}}_{SF}^B\right) \times \underline{\gamma}^N + \delta \underline{g}_P^N \\
& - \left(\Delta \underline{\omega}_{IN}^N + \Delta \underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \Delta \underline{v}^N \\
& - f_2 \left[\left(\hat{C}_B^N \, \hat{\underline{a}}_{SF}^B\right) \times \underline{\gamma}^N\right] \times \underline{\gamma}^N - f_1 \left(\hat{C}_B^N \, \delta \underline{a}_{SF}^B\right) \times \underline{\gamma}^N \\
& + \left(\Delta \underline{\omega}_{IN}^N + \Delta \underline{\omega}_{IE}^N\right) \times \Delta \underline{v}^N + f_2 \left[\left(\hat{C}_B^N \, \delta \underline{a}_{SF}^B\right) \times \underline{\gamma}^N\right] \times \underline{\gamma}^N
\end{aligned} \qquad \text{(A-11)}$$

Then substituting f_1 and f_2 from (A-9) and neglecting terms with third order error products, (A-11) becomes:

$$\begin{aligned}
\Delta \dot{\underline{v}}^N \approx {} & \hat{C}_B^N \, \delta \underline{a}_{SF}^B + \hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N + \delta \underline{g}_P^N \\
& - \left(\Delta \underline{\omega}_{IN}^N + \Delta \underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \Delta \underline{v}^N \\
& - \frac{1}{2} \left(\hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N\right) \times \underline{\gamma}^N - \left(\hat{C}_B^N \, \delta \underline{a}_{SF}^B\right) \times \underline{\gamma}^N + \left(\Delta \underline{\omega}_{IN}^N + \Delta \underline{\omega}_{IE}^N\right) \times \Delta \underline{v}^N
\end{aligned} \qquad \text{(A-12)}$$

Equation (A-12) is the error rate equation associated with the (2) velocity updating process.

APPENDIX B - HORIZONTAL VELOCITY ERROR RATE EQUATION DERIVATION

This appendix derives the horizontal form of the (A-12) general velocity error expression for the Equation (20) integrand.

The $\Delta \underline{\omega}_{IE}^{N}$ term in (A-12) is derived by expanding from the error free version of $\Delta \hat{\underline{\omega}}_{IE}^{N}$ in (9):

$$\begin{aligned}\underline{\omega}_{IE}^{N} &= C_{N*}^{N}\, \underline{\omega}_{IE}^{N*} = \hat{\underline{\omega}}_{IE}^{N} - \Delta \underline{\omega}_{IE}^{N} = \left(\hat{C}_{N*}^{N} - \Delta C_{N*}^{N}\right)\left(\hat{\underline{\omega}}_{IE}^{N*} - \delta \underline{\omega}_{IE}^{N*}\right) \\ &= \hat{C}_{N*}^{N}\, \hat{\underline{\omega}}_{IE}^{N*} - \Delta C_{N*}^{N}\, \hat{\underline{\omega}}_{IE}^{N*} - \left(\hat{C}_{N*}^{N} - \Delta C_{N*}^{N}\right) \delta \underline{\omega}_{IE}^{N*}\end{aligned} \tag{B-1}$$

Substituting $\hat{\underline{\omega}}_{IE}^{N}$ from (9) in (B-1) and neglecting the minor $\delta \underline{\omega}_{IE}^{N*}$ error in the reference derived data then finds

$$\Delta \underline{\omega}_{IE}^{N} = \Delta C_{N*}^{N}\, \hat{\underline{\omega}}_{IE}^{N*} \tag{B-2}$$

which also shows the $\hat{\underline{\omega}}_{IE}^{N}$ error to be of the large Δ type. Finally, because $\hat{C}_{N*}^{N}$ is a rotation around the vertical Z axis, ΔC_{N*}^{N} has no Z axis components, and (B-2) simplifies to

$$\Delta \underline{\omega}_{IE}^{N} = \Delta \underline{\omega}_{IE_H}^{N} = \Delta C_{N*}^{N}\, \hat{\underline{\omega}}_{IE_H}^{N*} \tag{B-3}$$

It can be shown similarly that

$$\Delta \underline{\omega}_{EN}^{N} = \Delta \underline{\omega}_{EN_H}^{N} = \Delta C_{N*}^{N}\, \hat{\underline{\omega}}_{EN_H}^{N*} \tag{B-4}$$

Recognizing that $\hat{\underline{\omega}}_{IN}^{N}$ satisfies $\hat{\underline{\omega}}_{IN}^{N} = \hat{\underline{\omega}}_{IE}^{N} + \hat{\underline{\omega}}_{EN}^{N}$ (and similarly for the true value), from (9) with (B-3) and (B-4), the error in $\hat{\underline{\omega}}_{IN}^{N}$ for (A-12) is

$$\begin{aligned}\Delta \underline{\omega}_{IN}^{N} &= \Delta \underline{\omega}_{IE}^{N} + \Delta \underline{\omega}_{EN}^{N} = \Delta \underline{\omega}_{IE_H}^{N} + \Delta \underline{\omega}_{EN_H}^{N} = \Delta \underline{\omega}_{IN_H}^{N} \\ &= \Delta C_{N*}^{N} \left(\hat{\underline{\omega}}_{IE_H}^{N*} + \hat{\underline{\omega}}_{EN_H}^{N*}\right) = \Delta C_{N*}^{N}\, \hat{\underline{\omega}}_{IN_H}^{N*}\end{aligned} \tag{B-5}$$

Neglecting the minor error in the reference data sets the error value in vertical velocity equation (7) to zero so that $\Delta\underline{v}^N$ in (A-12) becomes

$$\Delta\underline{v}^N = \Delta\underline{v}_H^N \tag{B-6}$$

Substituting the horizontal properties of (6) and (B-4) - (B-6) into (A-12), then finds for $\Delta\dot{\underline{v}}_H^N$ in (20):

$$\Delta\dot{\underline{v}}_H^N = \left[\begin{array}{c} \hat{C}_B^N\, \delta\underline{a}_{SF}^B + \hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N + \delta\underline{g}_P^N \\ + \underline{u}_Z^N \times \left[\hat{v}_{ZN}\left(\Delta\underline{\omega}_{IN_H}^N + \Delta\underline{\omega}_{IE_H}^N\right) - \left(\hat{\omega}_{IN_{ZN}} + \hat{\omega}_{IE_{ZN}}\right)\Delta\underline{v}_H^N\right] \\ - \frac{1}{2}\left(\hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N\right) \times \underline{\gamma}^N - \left(\hat{C}_B^N\, \delta\underline{a}_{SF}^B\right) \times \underline{\gamma}^N \end{array}\right]_H \tag{B-7}$$

Equation (B-7) is the horizontal velocity error rate integrated by (20) into the (15) observation equation.

APPENDIX C - ATTITUDE ERROR RATE EQUATION DERIVATION

The idealized error free form of the attitude rate in (2) is

$$\dot{C}_B^N = C_B^N\left(\underline{\omega}_{IB}^B \times\right) - \left(\underline{\omega}_{IN}^N \times\right) C_B^N \tag{C-1}$$

Substituting the (A-5) and (A-2) definitions into (C-1) yields:

$$\begin{aligned} \dot{C}_B^N - \delta\dot{C}_B^N &= \left(\hat{C}_B^N - \delta C_B^N\right)\left[\left(\tilde{\underline{\omega}}_{IB}^B - \delta\underline{\omega}_{IB}^B\right)\times\right] - \left[\left(\hat{\underline{\omega}}_{IN}^N - \Delta\underline{\omega}_{IN}^N\right)\times\right]\left(\hat{C}_B^N - \delta C_B^N\right) \\ &= \hat{C}_B^N\left(\tilde{\underline{\omega}}_{IB}^B \times\right) - \hat{C}_B^N\left(\delta\underline{\omega}_{IB}^B \times\right) - \delta C_B^N\left(\hat{\underline{\omega}}_{IB}^B \times\right) + \delta C_B^N\left(\delta\underline{\omega}_{IB}^B \times\right) \\ &\quad - \left(\hat{\underline{\omega}}_{IN}^N \times\right)\hat{C}_B^N + \left(\hat{\underline{\omega}}_{IN}^N \times\right)\delta C_B^N + \left(\Delta\underline{\omega}_{IN}^N \times\right)\hat{C}_B^N - \left(\Delta\underline{\omega}_{IN}^N \times\right)\delta C_B^N \end{aligned} \tag{C-2}$$

in which

$$\delta\underline{\omega}_{IB}^B \equiv \hat{\underline{\omega}}_{IB}^B - \underline{\omega}_{IB}^B \tag{C-3}$$

Substituting (2) for $\dot{\hat{C}}_B^N$ in (C-2) finds

$$\begin{aligned} \delta\dot{C}_B^N &= \hat{C}_B^N\left(\delta\underline{\omega}_{IB}^B \times\right) + \delta C_B^N\left(\tilde{\underline{\omega}}_{IB}^B \times\right) - \delta C_B^N\left(\delta\underline{\omega}_{IB}^B \times\right) \\ &\quad - \left(\hat{\underline{\omega}}_{IN}^N \times\right)\delta C_B^N - \left(\Delta\underline{\omega}_{IN}^N \times\right)\hat{C}_B^N + \left(\Delta\underline{\omega}_{IN}^N \times\right)\delta C_B^N \end{aligned} \tag{C-4}$$

The $\delta\dot{C}_B^N$ term in (C-4) is the derivative of (A-10) with (2) for $\dot{\hat{C}}_B^N$:

$$
\begin{aligned}
\delta\dot{C}_B^N = - &\left\{ \begin{matrix} f_1 \left(\dot{\underline{\gamma}}^N \times\right) + \dot{f}_1 \left(\underline{\gamma}^N \times\right) \\ + f_2 \dfrac{d}{dt}\left[\left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] + \dot{f}_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right) \end{matrix} \right\} \hat{C}_B^N \\
&- \left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \dot{\hat{C}}_B^N \\
= - &\left\{ \begin{matrix} f_1 \left(\dot{\underline{\gamma}}^N \times\right) + \dot{f}_1 \left(\underline{\gamma}^N \times\right) \\ + f_2 \dfrac{d}{dt}\left[\left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] + \dot{f}_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right) \end{matrix} \right\} \hat{C}_B^N \\
&- \left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \left[\hat{C}_B^N \left(\underline{\tilde{\omega}}_{IB}^B \times\right) - \left(\hat{\underline{\omega}}_{IN}^N \times\right) \hat{C}_B^N \right]
\end{aligned}
\tag{C-5}
$$

Then substituting (A-10) and (C-5) into (C-4) obtains after rearrangement:

$$
\begin{aligned}
& f_1 \left(\dot{\underline{\gamma}}^N \times\right) + \dot{f}_1 \left(\underline{\gamma}^N \times\right) \\
& + f_2 \frac{d}{dt}\left[\left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] + \dot{f}_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right) \\
& - \left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \left(\hat{\underline{\omega}}_{IN}^N \times\right) \\
= & - \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] - \left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] \\
& - \left(\hat{\underline{\omega}}_{IN}^N \times\right) \left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] + \left(\Delta\underline{\omega}_{IN}^N \times\right) \\
& + \left(\Delta\underline{\omega}_{IN}^N \times\right) \left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right]
\end{aligned}
\tag{C-6}
$$

From its definition, the cross-product matrix operator form of a vector is skew-symmetric (i.e., the element in row i column j equals the negative of the element in row j column i). It follows that the transpose of a cross-product operator matrix equals the negative of the matrix. Based on this property, the transpose of (C-6) is:

$$
\begin{aligned}
& - f_1 \left(\dot{\underline{\gamma}}^N \times\right) - \dot{f}_1 \left(\underline{\gamma}^N \times\right) \\
& + f_2 \frac{d}{dt}\left[\left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] + \dot{f}_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right) \\
& + \left(\hat{\underline{\omega}}_{IN}^N \times\right) \left[- f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \\
= & \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] + \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] \left[- f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \\
& + \left[- f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \left(\hat{\underline{\omega}}_{IN}^N \times\right) - \left(\Delta\underline{\omega}_{IN}^N \times\right) \\
& - \left[- f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \left(\Delta\underline{\omega}_{IN}^N \times\right)
\end{aligned}
\tag{C-7}
$$

Subtracting (C-7) from (C-6), dividing by 2, and rearranging obtains

$$
\begin{aligned}
f_1 \left(\dot{\underline{\gamma}}^N \times\right) = &- \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times\right] + \left(\Delta\underline{\omega}_{IN}^N \times\right) \\
&+ f_1 \left[\left(- \hat{\underline{\omega}}_{IN}^N + \frac{1}{2} \Delta\underline{\omega}_{IN}^N + \frac{1}{2} \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times\right] \left(\underline{\gamma}^N \times\right) \\
&- f_1 \left(\underline{\gamma}^N \times\right) \left[\left(- \hat{\underline{\omega}}_{IN}^N + \frac{1}{2} \Delta\underline{\omega}_{IN}^N + \frac{1}{2} \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times\right] \\
&+ \frac{1}{2} f_2 \left(\underline{\gamma}^N \times\right) \left(\underline{\gamma}^N \times\right) \left[\left(\Delta\underline{\omega}_{IN}^N - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times\right] \\
&+ \frac{1}{2} f_2 \left[\left(\Delta\underline{\omega}_{IN}^N - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times\right] \left(\underline{\gamma}^N \times\right) \left(\underline{\gamma}^N \times\right) - \dot{f}_1 \left(\underline{\gamma}^N \times\right)
\end{aligned}
\tag{C-8}
$$

or, after applying [3 - (3.1.1-22)]:

$$
\begin{aligned}
f_1 \left(\dot{\underline{\gamma}}^N \times\right) = &- \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times\right] + \left(\Delta\underline{\omega}_{IN}^N \times\right) \\
&+ f_1 \left\{\left[\left(- \hat{\underline{\omega}}_{IN}^N + \frac{1}{2} \Delta\underline{\omega}_{IN}^N + \frac{1}{2} \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times \underline{\gamma}^N\right] \times\right\} \\
&+ \frac{1}{2} f_2 \left(\underline{\gamma}^N \times\right) \left(\underline{\gamma}^N \times\right) \left[\left(\Delta\underline{\omega}_{IN}^N - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times\right] \\
&+ \frac{1}{2} f_2 \left[\left(\Delta\underline{\omega}_{IN}^N - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times\right] \left(\underline{\gamma}^N \times\right) \left(\underline{\gamma}^N \times\right) - \dot{f}_1 \left(\underline{\gamma}^N \times\right)
\end{aligned}
\tag{C-9}
$$

But from (A-9) and (C-9),

$$
f_1 = 1 - \frac{\gamma^2}{3\,!} + \cdots = \text{order of } \gamma^2 \qquad \dot{f}_1 = - \frac{\gamma}{3} \dot{\gamma} + \cdots = \text{order of } \gamma \, \Delta\omega_{IN}
\tag{C-10}
$$

Hence,

$$
\begin{aligned}
\left(\dot{\underline{\gamma}}^N \times\right) = &- \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times\right] + \left(\Delta\underline{\omega}_{IN}^N \times\right) \\
&+ \left\{\left[\left(- \hat{\underline{\omega}}_{IN}^N + \frac{1}{2} \Delta\underline{\omega}_{IN}^N + \frac{1}{2} \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times \underline{\gamma}^N\right] \times\right\} \\
&+ \text{order of } \gamma^2 \, \Delta\omega_{IN} + \text{order of } \gamma^2 \, \delta\omega_{IB}
\end{aligned}
\tag{C-11}
$$

Therefore,

$$
\dot{\underline{\gamma}}^N = - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B - \left(\hat{\underline{\omega}}_{IN}^N - \frac{1}{2} \Delta\underline{\omega}_{IN}^N - \frac{1}{2} \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right) \times \underline{\gamma}^N + \Delta\underline{\omega}_{IN}^N + \cdots
\tag{C-12}
$$

or with (B-5):

$$\dot{\underline{\gamma}}^N \approx - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}^N + \Delta\underline{\omega}_{IN_H}^N + \frac{1}{2}\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B + \Delta\underline{\omega}_{IN_H}^N\right) \times \underline{\gamma}^N \quad \text{(C-13)}$$

Equation (C-13) is the error rate equation associated with the (2) attitude updating processes performed in the INS computer.

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