COARSE LEVELING OF INS ATTITUDE UNDER DYNAMIC TRAJECTORY CONDITIONS

Paul G. Savage
Strapdown Associates, Inc.

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www.strapdownassociates.com
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ABSTRACT

Approximate attitude initialization is required by an inertial navigation system (INS) to initiate fine alignment Kalman filter operations. This article describes a simple approach for setting strapdown INS initial attitude to an approximately correct roll/pitch attitude from horizontal (Coarse Leveling), but with arbitrary and unknown heading relative to north. Coarse Leveling attitude initialization is compatible with recent analytical findings showing how linearized fine alignment Kalman filters can be structured for small initial roll/pitch errors and a wide angle initial heading error. The Coarse Leveling process described requires velocity from a GPS receiver (or the equivalent), requires no special maneuvers, but does require maintaining an approximately steady horizontal velocity during the short Coarse Leveling time period. The method is designed for dynamic trajectory conditions, allowing for attitude rotation during Coarse Leveling and separation of the INS from the GPS antenna. Examples demonstrate accuracy versus leveling time under variations from steady horizontal velocity.

INTRODUCTION

Coarse Alignment of attitude in a strapdown INS computer establishes a set of attitude parameters with small angular errors in all axes. This has been a general initialization requirement for ensuing Kalman filter fine alignment operations for compatibility with traditional linearized Kalman/system error modeling. Recent analytical findings have reformulated linearized Kalman filter error models to be compatible with a wide angle initial heading error, requiring small initial angular errors to be only relative to the local horizontal (i.e., roll and pitch). Thus, for Kalman filters designed using the reformulated approach, only Coarse Leveling (approximate attitude determination relative to the horizontal) is required for Kalman fine alignment initialization, thereby simplifying requirements on the initialization process.

When a separate attitude reference device is available, Coarse Leveling becomes a simple matter of transferring the reference attitude into the INS while accounting for known mounting orientation differences between the two devices. When a separate attitude reference device is not available, this article describes a simple and rapid Coarse Leveling method using velocity data from a GPS receiver (or the equivalent).

The Coarse Leveling approach presented is an extension of the method described in [1 - Sect. 6.1.1], a simple and rapid procedure for performing strapdown INS Coarse Alignment using INS strapdown accelerometer measurements. Reference [1 - Sect. 6.1.2] shows how for stationary conditions, a linearized fine alignment Kalman filter can be configured for wide initial heading angle error, thereby allowing Coarse Leveling only for attitude initialization. Reference [2] expands on the [1 - Sect. 6.1.2] Kalman modeling approach to non-stationary applications having a large initial heading error, thereby enabling general Kalman alignment initialization with only Coarse Leveling. The [2] measurement equation is based on the revised formulation of [3] designed to reduce the impact of second order initial leveling errors on Kalman estimation accuracy. As a result, Coarse Leveling initialization accuracy requirements are also reduced for a [2] type Kalman configuration.

This article extends the stationary Coarse Leveling method of [1] for operation under dynamic trajectory conditions with the new [2] type wide-heading-error fine alignment Kalman filter. The new Coarse Alignment method does not depend on special maneuvering, does require velocity inputs from a GPS receiver (or equivalent), and is based on maintaining an approximately steady horizontal velocity during the Coarse Leveling time period (this restriction becomes less important for slowly moving vehicles, e.g., ships). Examples demonstrate Coarse Leveling accuracy versus leveling time as a function of horizontal velocity deviation from steady motion.

The mathematical notation used in this article is the same as in [1]:

$\underline{V}$ = Vector without specific coordinate frame designation. A vector is a parameter that has length and direction. Vectors used in this article are classified as “free vectors”, hence, have no preferred location in coordinate frames in which they are analytically described.

$|\underline{V}|$ = Magnitude of vector $\underline{V}$.

$\underline{V}^A$ = Column matrix with elements equal to the projection of $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit vector parallel to that coordinate axis.

$\left(\underline{V}^A \times\right)$ = Skew symmetric (or cross-product) form of $\underline{V}^A$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{XA}, V_{YA}, V_{ZA} are the components of $\underline{V}^A$. The matrix product of $\left(\underline{V}^A \times\right)$ with another A frame vector equals the cross-product of $\underline{V}^A$ with the vector in the A frame, i.e.: $\left(\underline{V}^A \times\right) \underline{W}^A = \underline{V}^A \times \underline{W}^A$.

$C_{A_2}^{A_1}$ = Direction cosine matrix that transforms a vector from its coordinate frame A_2 projection form to its coordinate frame A_1 projection form, i.e.: $\underline{V}^{A_1} = C_{A_2}^{A_1} \, \underline{V}^{A_2}$.

The columns of $C_{A_2}^{A_1}$ are projections on $A1$ axes of unit vectors parallel to $A2$ axes. Conversely, the rows of $C_{A_2}^{A_1}$ are projections on $A2$ axes of unit vectors parallel to $A1$ axes. An important property of $C_{A_2}^{A_1}$ is that its inverse equals its transpose.

$\dot{(\;)} = \dfrac{d(\;)}{dt}$ = Time derivative of ( ).

FUNDAMENTAL COARSE LEVELING CONCEPT

Specific force generated acceleration (sometimes denoted as "total acceleration minus gravity"), is acceleration relative to gravitational inertial space, and is directly measurable by accelerometers [4]. The Coarse Leveling approach presented in this article uses integrated specific force as the basis for attitude determination relative to the horizontal. Integrated specific force is calculated within the INS by transforming the INS accelerometer output vector into a non-rotating inertial coordinate reference frame, integrating, correcting for displacement of the INS from the GPS antenna under rotation, and transforming the result back to the current sensor frame. The matrix used for transformation is calculated as an integration operation using INS strapdown gyro angular rates. When it is determined from GPS receiver horizontal velocity data that the INS calculated integrated specific force is approximately vertical, it is used to find sensor frame angular orientation from horizontal, thereby completing Coarse Leveling.

ANALYTICAL BASIS

The analytical basis for the Coarse Leveling approach is the revised Newtonian law of natural motion between two points introduced in [4]:

$$\frac{d^2}{dt^2}\underline{r}_{2,1}^{I} = \underline{a}_{SF_2}^{I} - \underline{a}_{SF_1}^{I} + \Delta\underline{g}_{2,1}^{I} \qquad (1)$$

where

1, 2 = Designation for parameters at location points 1 and 2.

$\underline{r}_{2,1}$ = Relative position vector from point 1 to point 2.

$\underline{a}_{SF_1}$, $\underline{a}_{SF_2}$ = Specific force of points 1 and 2 (measurable by accelerometers).

$\Delta\underline{g}_{2,1}$ = Difference in gravity between points 1 and 2 (gravity at 2 minus gravity at 1).

I = Inertially non-rotating coordinate frame as defined in [4].

The integral of (1) over the time period for Coarse Leveling is

$$\begin{aligned} &\frac{d}{dt}\underline{r}_{2,1}^{I} - \left(\frac{d}{dt}\underline{r}_{2,1}^{I}\right)_0 = \underline{\upsilon}_2^I - \underline{\upsilon}_1^I + \int_0^t \Delta\underline{g}_{2,1}^I \, dt \\ &\underline{\upsilon}_2^I \equiv \int_0^t \underline{a}_{SF\,2}^I \, dt \qquad \underline{\upsilon}_1^I \equiv \int_0^t \underline{a}_{SF\,1}^I \, dt \end{aligned} \tag{2}$$

where

0 = Subscript indicating the value of the designated parameter at the start of Coarse Leveling.

t = Time from the start of Coarse Leveling.

$\underline{\upsilon}_1$, $\underline{\upsilon}_2$ = Integrated specific force vector at locations 1 and 2 where the integral is taken in inertially non-rotating coordinate frame I .

For the Coarse Leveling approach presented in this article, (2) will be used to describe integrated specific force measured within the INS and motion of the GPS antenna measured by the GPS receiver.

CALCULATING INTEGRATED SPECIFIC FORCE WITH INS INERTIAL SENSOR DATA

Within the INS computer, the relevant parameters in (2) are defined as follows:

B = Rotating coordinate frame aligned with INS strapdown inertial sensor axes.

B_0 = Inertially non-rotating coordinates for the I frame in (2), aligned with the B frame at the start of Coarse Leveling.

Location 1 = The GPS antenna position location.

Location 2 = The INS position location.

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With the above definitions, the difference in gravity $\Delta\underline{g}_{2,1}$ between locations 1 and 2 is virtually zero, and (2) rearranged becomes:

$$\underline{\upsilon}_{GPS}^{B_0} = \underline{\upsilon}_{INS}^{B_0} - \frac{d}{dt}\underline{r}_{INS,GPS}^{B_0} + \left(\frac{d}{dt}\underline{r}_{INS,GPS}^{B_0}\right)_0 \tag{3}$$

where

INS, GPS = Subscript designations for parameter values at the INS and the GPS antenna locations.

The $\underline{\upsilon}_{GPS}^{B_0}$ vector in (3) is the integrated specific force vector in non-rotating B_0 frame inertial coordinates. The equivalent in B frame coordinates is obtained from the transformation:

$$\underline{\upsilon}_{GPS}^{B} = \left(C_B^{B_0}\right)^T \underline{\upsilon}_{GPS}^{B_0} \tag{4}$$

with $C_B^{B_0}$ calculated as an integration process since the start of Coarse Leveling using the classic form [1 - Eq. (3.3.2-9)]:

$$C_B^{B_0} = I + \int_0^t \dot{C}_B^{B_0}\, dt \qquad \dot{C}_B^{B_0} = C_B^{B_0}\left(\underline{\omega}^B \times\right) \tag{5}$$

where

$\underline{\omega}$ = INS angular rate vector relative to non-rotating inertial space [4], measurable by the INS strapdown gyros as $\underline{\omega}^B$.

I = The identity matrix.

The $\underline{\upsilon}_{INS}^{B_0}$ integrated specific force vector in (3) is calculated with (2) using specific force measured by INS accelerometers in the B frame and transformed to the B_0 frame using $C_B^{B_0}$ from (5):

$$\underline{\upsilon}_{INS}^{B_0} = \int_0^t C_B^{B_0}\, \underline{a}_{SF_{INS}}^{B}\, dt \tag{6}$$

where

$\underline{a}_{SF_{INS}}$ = Specific force acceleration of the INS location, measurable in the B frame by INS strapdown accelerometers as $\underline{a}_{SF_{INS}}^{B}$.

The $\frac{d}{dt}\underline{r}_{INS,GPS}^{B_0}$ vector in (3) is calculated using (5) and the assumption that $\underline{r}_{INS,GPS}$ is fixed in the B frame:

$$\begin{aligned}\frac{d}{dt}\underline{r}_{INS,GPS}^{B_0} &= \frac{d}{dt}\left(C_B^{B_0}\, \underline{r}_{INS,GPS}^{B}\right) = \dot{C}_B^{B_0}\, \underline{r}_{INS,GPS}^{B} + C_B^{B_0} \frac{d}{dt}\underline{r}_{INS,GPS}^{B} \\ &= C_B^{B_0}\left(\underline{\omega}^B \times \underline{r}_{INS,GPS}^{B}\right)\end{aligned} \tag{7}$$

Then, for the $\frac{d}{dt}\underline{r}_{INS,GPS}^{B_0}$ terms in (3):

$$-\frac{d}{dt}\underline{r}^{B_0}_{INS,GPS}+\left(\frac{d}{dt}\underline{r}^{B_0}_{INS,GPS}\right)_0=-C^{B_0}_B\left(\underline{\omega}^B\times\underline{r}^B_{INS,GPS}\right)+\left[C^{B_0}_B\left(\underline{\omega}^B\times\underline{r}^B_{INS,GPS}\right)\right]_0$$

$$=\underline{\omega}^{B_0}_0\times\underline{r}^B_{INS,GPS}-C^{B_0}_B\left(\underline{\omega}^B\times\underline{r}^B_{INS,GPS}\right) \tag{8}$$

Substituting (8) into (3) and the result into (4), finds for $\underline{\upsilon}^B_{GPS}$:

$$\underline{\upsilon}^B_{GPS}=\left(C^{B_0}_B\right)^T\underline{\upsilon}^{B_0}_{INS}-\left[\underline{\omega}^B-\left(C^{B_0}_B\right)^T\underline{\omega}^{B_0}_0\right]\times\underline{r}^B_{INS,GPS} \tag{9}$$

The normalized version of (9) will be used to complete Coarse Leveling when it has been determined that $\underline{\upsilon}_{GPS}$ is approximately vertical:

$$\underline{u}^B_{\upsilon GPS}=\underline{\upsilon}^B_{GPS}/\left|\underline{\upsilon}^B_{GPS}\right| \tag{10}$$

where

$\underline{u}^B_{\upsilon GPS}$ = Unit vector along $\underline{\upsilon}^B_{GPS}$.

CALCULATING INTEGRATED SPECIFIC FORCE WITH GPS VELOCITY DATA

At the GPS antenna location, the relevant parameters in (2) are defined as follows:

N = Locally level navigation coordinate frame used by the GPS receiver to provide velocity data to the INS. For Coarse Leveling, the approximation is made that the *N* frame is inertially non-rotating (i.e., by neglecting earth's rotation and INS translational motion over the earth that slowly rotate the local vertical), hence, *N* can be used in (2) as an *I* frame.

Location 1 = The center of the earth.

Location 2 = The GPS antenna.

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From the above definition, recognizing that the specific force at earth's center is zero [4], and approximating $\Delta\underline{g}^N_{GPS,Ecntr}$ as constant over the short Coarse Leveling time period, (2) becomes:

$$\Delta\underline{v}^N_{GPS,Ecntr}\equiv\underline{v}^N_{GPS,Ecntr}-\underline{v}^N_{GPS,Ecntr_0}=\underline{\upsilon}^N_{GPS}+\Delta\underline{g}^N_{GPS,Ecntr}\,t \tag{11}$$

where

GPS, Ecntr = Subscripts designating parameter values at the GPS antenna location and at the center of the earth.

$\underline{v}_{GPS,Ecntr}$ = Velocity relative to the earth measured by the GPS receiver.

Assuming that only gravity created by earth's mass impacts motion, and because earth's gravity is zero at earth's center [4], (11) becomes

$$\Delta\underline{v}_{GPS,Ecntr}^{N} = \Delta\underline{v}_{GPS,Ecntr_H}^{N} + \Delta v_{GPS,Ecntr_{Up}}\,\underline{u}_{Up}^{N} = \underline{\upsilon}_{GPS}^{N} - g\,t\,\underline{u}_{Up}^{N} \qquad (12)$$

where

H = Subscript indicating the horizontal component of the designated parameter.

$\underline{u}_{Up}$ = Unit vector upward.

g = Magnitude of gravity created by earth's mass.

With rearrangement, (12) assumes the desired form:

$$\underline{\upsilon}_{GPS}^{N} = \Delta\underline{v}_{GPS,Ecntr_H}^{N} + \left(\Delta v_{GPS,Ecntr_{Up}} + g\,t\right)\underline{u}_{Up}^{N} \qquad (13)$$

Equation (13) shows that integrated specific force is vertical at the antenna location when the horizontal component of GPS receiver velocity $\Delta\underline{v}_{GPS,Ecntr_H}^{N}$ is zero. Thus, when the magnitude of $\Delta\underline{v}_{GPS,Ecntr_H}^{N}$ is small, the *B* frame integrated specific force $\underline{\upsilon}_{GPS}^{B}$ from (4) can be considered approximately vertical, hence, can be used to complete Coarse Leveling operations.

The normalized version of (13) will be used to determine when integrated specific force is vertical:

$$\underline{u}_{\upsilon GPS}^{N} = \underline{\upsilon}_{GPS}^{N} / \left|\underline{\upsilon}_{GPS}^{N}\right| \qquad (14)$$

where

$\underline{u}_{\upsilon GPS}^{N}$ = Unit vector along $\underline{\upsilon}_{GPS}^{N}$.

DETERMINING WHEN INTEGRATED SPECIFIC FORCE IS VERTICAL

Coarse leveling is completed when integrated specific force has been determined to be approximately vertical based on the angular orientation of $\underline{u}_{\upsilon GPS}^{N}$ in (14) relative to $\underline{u}_{Up}^{N}$:

$$\underline{\gamma}^{N} = \underline{u}_{Up}^{N} \times \underline{u}_{\upsilon GPS}^{N} \qquad (15)$$

where

$\underline{\gamma}^{N}$ = Angular deviation of $\underline{u}^{N}_{\upsilon GPS}$ from vertical.

Coarse Leveling initializes *B* frame attitude relative to the apparent vertical as represented by $\underline{u}_{\upsilon GPS}$, the normalized integrated specific force vector. Thus, $\underline{\gamma}^{N}$ represents the leveling error that would result when using $\underline{u}_{\upsilon GPS}$ to represent verticality. Substituting (14) with (13) into (15) gives $\underline{\gamma}^{N}$ as a function of GPS measured horizontal velocity change $\Delta \underline{v}^{N}_{GPS,Ecntr_H}$:

$$\underline{\gamma}^{N} = \underline{u}^{N}_{Up} \times \Delta \underline{v}^{N}_{GPS,Ecntr_H} / \left| \underline{\upsilon}^{N}_{GPS} \right| \tag{16}$$

The magnitude of (16) rearranged is used to establish maximum values on $\Delta \underline{v}^{N}_{GPS,Ecntr_H}$ for $\underline{\gamma}^{N}$ to be within acceptable accuracy:

$$\left| \Delta \underline{v}^{N}_{GPS,Ecntr_H} \right| < \left| \underline{\upsilon}^{N}_{GPS} \right| \gamma_{Lmt} \tag{17}$$

where

γ_{Lmt} = Magnitude of the allowable variation of $\underline{u}_{\upsilon GPS}$ from vertical to meet Coarse Leveling accuracy requirements.

Satisfying (17) assures that $\underline{u}_{\upsilon GPS}$ will be approximately vertical (within acceptable limits), thereby allowing Coarse Leveling to complete with assurance that the leveling error will not exceed the γ_{Lmt} requirement.

COMPLETING COARSE LEVELING

Having satisfied (17), the INS measured *B* frame components of $\underline{u}_{\upsilon GPS}$ (i.e., $\underline{u}^{B}_{GPS}$) can be interpreted to be a unit vector along the vertical, and used to erect a direction cosine matrix C^{N*}_{B} to complete Coarse Leveling:

$$\underline{u}^{B}_{ZN*} \approx \underline{u}^{B}_{\upsilon GPS} \tag{18}$$

where

*N** = Locally level navigation coordinate frame at arbitrary heading relative to the GPS velocity input *N* frame, and with the third axis (*Z*) parallel to the *N* frame Z-axis along the upward vertical.

$\underline{u}_{ZN*}$ = Unit vector along the $N*$ frame Z-axis equal to $\underline{u}_{ZN*}^{B}$ in the *B* frame and $\underline{u}_{ZN*}^{N*} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ in the *N** frame.

Initializing C_B^{N*} is based on the fundamental transformation property:

$$\underline{u}_{ZN*}^{B} = \left(C_B^{N*}\right)^T \underline{u}_{ZN*}^{N*} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} \tag{19}$$

where

$C_{i\,j}$ = Element in row i, column j of C_B^{N*}.

Equation (19) shows that $\underline{u}_{ZN*}^{B}$ equals the transposed third row of C_B^{N*}. It can be verified that the roll/pitch orientation of the *B* frame relative to the horizontal is a function of only C_B^{N*} third row elements [1 - Eq. (4.1.2-1)]. Thus, setting the third row of C_B^{N*} to the transpose of $\underline{u}_{ZN*}^{B}$ from (18) will assure a C_B^{N*} matrix with the correct angular orientation of the *B* frame relative to the horizontal (i.e., the *N** in C_B^{N*} would be correctly leveled).

With the third row of C_B^{N*} established by (18) and (19), the remaining two rows can be arbitrarily selected, providing that direction cosine matrix orthogonality/normality constraints are satisfied: The magnitude of each row must be unity, and the dot product between any two rows (treated as vectors) must be zero. Reference [1 - Sect. 6.1.1] provides a simple example of setting the second row of C_B^{N*} whereby the first element is set to zero, and the remaining two elements are set so the dot product with row three is zero:

$$C_{21}\,C_{31} + C_{22}\,C_{32} + C_{23}\,C_{33} = C_{22}\,C_{32} + C_{23}\,C_{33} = 0 \tag{20}$$

Equation (20) is satisfied if

$$C_{21} = 0 \qquad C_{22} = K\,C_{33} \qquad C_{23} = -K\,C_{32} \tag{21}$$

with the constant K selected to meet the second row normalization requirement:

$$C_{22}^2 + C_{23}^2 = K^2\left(C_{33}^2 + C_{32}^2\right) = 1 \quad \rightarrow \quad K = 1/\sqrt{C_{33}^2 + C_{32}^2} \tag{22}$$

Similar to (19), the second row of C_B^{N*} transposed is $\underline{u}_{YN*}^{B}$, a unit vector along the *N** frame *Y*-axis projected on *B* frame axes, or with (21) and (22):

$$\underline{u}_{YN*}^{B} = \begin{bmatrix} 0 \\ -C_{33} / \sqrt{C_{32}^{2} + C_{33}^{2}} \\ C_{32} / \sqrt{C_{32}^{2} + C_{33}^{2}} \end{bmatrix} \tag{23}$$

Since unit vectors along coordinate frame axes are perpendicular, $\underline{u}_{XN*}^{B}$, the *B* frame projection of a unit vector along the *N** frame *X*-axis, can be calculated as the cross-product of $\underline{u}_{YN*}^{B}$ with $\underline{u}_{ZN*}^{B}$:

$$\underline{u}_{XN*}^{B} = \underline{u}_{YN*}^{B} \times \underline{u}_{ZN*}^{B} \tag{24}$$

Analogous to (19), $\underline{u}_{XN*}^{B}$ and $\underline{u}_{YN*}^{B}$ equal the first and second rows of C_{B}^{N*} transposed. Thus, having found $\underline{u}_{XN*}^{B}$, $\underline{u}_{YN*}^{B}$, and $\underline{u}_{ZN*}^{B}$ with (18), (23) and (24), Coarse Leveling is completed by setting C_{B}^{N*} to

$$C_{B}^{N*} = \left[\underline{u}_{XN*}^{B} \;\; \underline{u}_{YN*}^{B} \;\; \underline{u}_{ZN*}^{B} \right]^{T} \tag{25}$$

MINUMUM ALLOWABLE TIME TO COMPLETE COARSE LEVELING

Under a particular dynamic trajectory condition, the $\left| \Delta \underline{v}_{GPS,Ecntr_{H}}^{N} \right|$ velocity change in (17) might initially grow slower than $\left| \underline{\upsilon}_{GPS}^{N} \right| \gamma_{Lmt}$, producing immediate Coarse Leveling completion by the (17) test. However, if $\underline{u}_{GPS}^{B}$ used for C_{B}^{N*} determination is in error, a leveling error could then be created that exceeds γ_{Lmt} requirements:

$$\hat{\underline{u}}_{\upsilon GPS}^{B} \equiv \underline{u}_{\upsilon GPS}^{B} + \delta \underline{u}_{\upsilon GPS}^{B} \qquad \delta \underline{\gamma}^{B} \times \underline{u}_{\upsilon GPS}^{B} = \hat{\underline{u}}_{\upsilon GPS}^{B} \quad \rightarrow \quad \delta \underline{\gamma}^{B} = \underline{u}_{\upsilon GPS}^{B} \times \delta \underline{u}_{\upsilon GPS}^{B} \tag{26}$$

where

$\hat{\underline{u}}_{\upsilon GPS}^{B}$ = INS computed $\underline{u}_{\upsilon GPS}^{B}$ containing error.

$\delta \underline{u}_{\upsilon GPS}^{B}$ = Error in $\hat{\underline{u}}_{\upsilon GPS}^{B}$.

$\delta \underline{\gamma}^{B}$ = Additional leveling error generated by $\delta \underline{u}_{\upsilon GPS}^{B}$ (i.e., in addition to the $\underline{\gamma}^{B}$ leveling created by $\underline{u}_{\upsilon GPS}^{B}$ not being exactly vertical).

From (10),

$$\delta \underline{u}_{\upsilon GPS}^{B} = \delta \underline{\upsilon}_{GPS}^{B} / \left| \underline{\upsilon}_{GPS}^{B} \right| - \left(\underline{\upsilon}_{GPS}^{B} / \left| \underline{\upsilon}_{GPS}^{B} \right|^{2} \right) \delta \left| \underline{\upsilon}_{GPS}^{B} \right| \tag{27}$$

where

$\delta \underline{\upsilon}_{GPS}^{B}$ = INS error in calculating the $\underline{\upsilon}_{GPS}^{B}$ that created $\delta \underline{\gamma}^{B}$.

For small velocity changes, (13) shows that $\left| \underline{\upsilon}_{GPS}^{N} \right|$ will approximately equal g t. With this approximation, substitution of (27) into (26) finds

$$\delta \underline{\gamma}^{B} = \underline{u}_{\upsilon GPS}^{B} \times \delta \underline{\upsilon}_{GPS}^{B} / (g\,t) \tag{28}$$

Equation (28) shows that a $\delta \underline{\upsilon}_{GPS}^{B}$ error can produce a very large additional leveling error $\delta \underline{\gamma}^{B}$ if Coarse Leveling is completed by (17) when t is small. If the error adds in magnitude to $\underline{\upsilon}_{GPS}^{B}$ when Coarse Leveling is terminated, the resulting C_{B}^{N*} matrix could have an unacceptably large leveling error. For example, consider the effect of structural bending between the INS and GPS antenna under dynamic trajectory conditions. This effect was ignored in (7) when $\frac{d}{dt} \underline{r}_{INS,GPS}^{B}$ was approximated as zero, thus leading to an error of $\frac{d}{dt} \underline{r}_{INS,GPS}^{B}$ in (9) for the INS calculation of $\underline{\upsilon}_{GPS}^{B}$. If a 1 Hz bending mode oscillation is excited to a 0.1 ft amplitude by severe trajectory dynamics, the resulting $\frac{d}{dt} \underline{r}_{INS,GPS}^{B}$ will generate a $\delta \underline{\upsilon}_{GPS}^{B}$ error of amplitude 0.1 ft × 1 Hz × 2 π = 0.6 fps. If (17) completes Coarse Leveling at 0.2 sec, (28) shows that an additional leveling error $\delta \underline{\gamma}^{B}$ would be generated of 0.6 / (32.2 × 0.2) = 0.093 rad = 5.3 deg, significantly larger than a typical γ_{Lmt} setting. On the other hand, if Coarse Leveling completion was delayed to 1.5 sec, (28) shows that with the same 0.6 fps $\delta \underline{\upsilon}_{GPS}^{B}$ error, $\delta \underline{\gamma}^{B}$ would equal 0.6 / (32.2 × 1.5) = 0.012 rad = 0.71 deg (small enough to be ignored when compared, for example, with a 3 deg γ_{Lmt} requirement).

The previous discussion demonstrates the need to set a minimum time for Coarse Leveling. A more detailed system error analysis is required to determine the minimum time setting for a particular application.

TIME INTERVAL VERSUS ACCURACY FOR COARSE LEVELING

The time interval for Course Leveling is implicitly determined to be when (17) is satisfied, depending on the magnitude of integrated specific force $\left|\underline{\upsilon}_{GPS}^{N}\right|$ of the GPS antenna and the specified leveling accuracy γ_{Lmt}. For example, (13) shows that for small velocity changes, $\left|\underline{\upsilon}_{GPS}^{N}\right|$ will approximately equal g t. Substitution in (17) shows with rearrangement that γ_{Lmt} leveling accuracy can be achieved when $t = \left|\Delta \underline{v}_{GPS,Ecntr_H}^{N}\right| / \left(g \; \gamma_{Lmt}\right)$. Using this formula (and assuming that the minimum leveling time setting discussed in the previous section is smaller than t), Table 1 illustrates the time t required for a specified leveling accuracy γ_{Lmt} as a function of the GPS antenna horizontal velocity change $\left|\Delta \underline{v}_{GPS,Ecntr_H}^{N}\right|$ determined from GPS velocity data.

Table 1 - Coarse Leveling Time For Specified Leveling Accuracy And Velocity Change

	GPS Antenna Horizontal Velocity Change			
	5 fps	10 fps	20 fps	50 fps
Time For 3 Deg Leveling Accuracy (sec)	3.0	5.9	12.9	29.7
Time For 2 Deg Leveling Accuracy (sec	4.4	8.9	18.8	44.5
Time For 1 Deg Leveling Accuracy (sec)	8.9	17.8	35.6	89.0

For proper interpretation, Table 1 should be viewed in context with the trajectory profiles shown in Fig. 1.

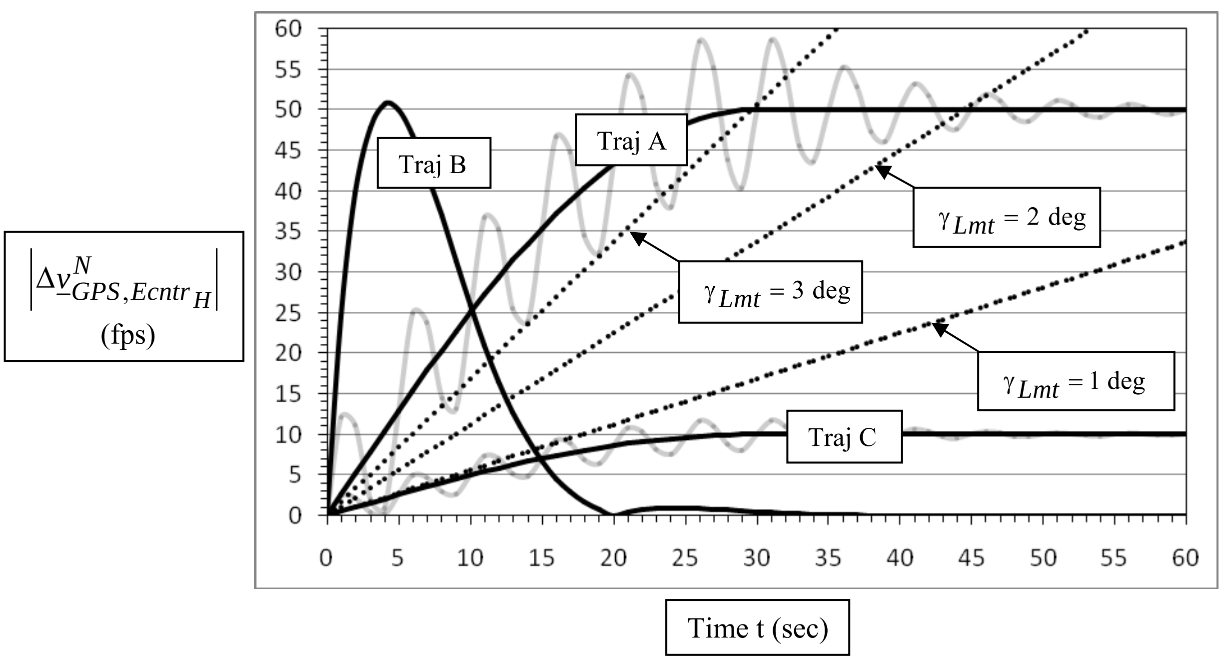

Fig. 1 - Integrated Specific Force Profiles

Fig. 1 illustrates integrated specific force $\left|\Delta \underline{v}^{N}_{GPS,Ecntr_H}\right|$ time histories for three trajectory profiles (Traj A, B, and C). Shown dotted in Figure 1 for $\left|\underline{\upsilon}^{N}_{GPS}\right| \approx g\, t$, are (17) test conditions when $\left|\Delta \underline{v}^{N}_{GPS,Ecntr_H}\right| = \left|\underline{\upsilon}^{N}_{GPS}\right| \gamma_{Lmt}$ for each of the three Table 1 γ_{Lmt} settings (1, 2 and 3 deg). The leveling times listed in Table 1 correspond to the intersection of the γ_{Lmt} dotted lines with horizontal lines at the corresponding Table 1 velocity change values (e.g., the 2 deg γ_{Lmt} line intersects the horizontal 20 fps $\left|\Delta \underline{v}^{N}_{GPS,Ecntr_H}\right|$ line at the Table 1 leveling time of 17.8 sec.)

For each trajectory in Fig. 1, Coarse Leveling completion would be initiated by the (17) test at the earliest time a particular trajectory is below the specified γ_{Lmt} dotted accuracy line. If the initial trajectory slope is greater than the specified γ_{Lmt} slope, Coarse Leveling would complete when the trajectory first crosses the γ_{Lmt} line "from above". If the initial trajectory slope is less than the specified γ_{Lmt} slope, Coarse Leveling would theoretically complete immediately (at time 0 +). However, as discussed in the previous section, this can cause a problem in the presence of errors in the calculated $\underline{u}^{B}_{\upsilon_{GPS}}$, and is avoided by setting a minimum time for Coarse Leveling completion. The remainder of this section will discuss the leveling time required for the former case (crossing the γ_{Lmt} line "from above") for each Fig. 1 trajectory, compared with the Table 1 Coarse Leveling time values. The discussion assumes that the vehicle is holding an approximate steady course (e.g., approximate steady magnetic heading at constant throttle setting) with no attempt to actively maintain a steady GPS indicated velocity (i.e., by steering and throttle control). It should be noted during the discussion, however, that shorter leveling times can also be directly obtained by manually or automatically controlling the vehicle to an approximately steady GPS based horizontal velocity, thus preventing large horizontal velocity changes from ever occurring.

First consider the 29.7 sec leveling time shown in Table 1 for a 50 fps velocity change and 3 deg γ_{Lmt} accuracy requirement. This might correspond to the black lined trajectory A case in Fig. 1 representing, for example, an aircraft moving into a region of changed horizontal air movement (e.g., through a weather front). Assuming that the aircraft is not being controlled to maintain steady GPS velocity, the changed air velocity would gradually accelerate the aircraft until it was at the same velocity relative to the air mass as it had before entering the new air movement region. As a result, the aircraft would change its velocity relative to the ground by the change in air velocity (by 50 fps in Fig. 1), eventually crossing the 3 deg γ_{Lmt} line ("from above") at 29.7 sec, thus completing Coarse Leveling by the (17) test. But in an actual case, trajectory A might include the effects of severe air turbulence, producing short term motion changes of the GPS antenna, hence, generating additional velocity transients (10 fps gray oscillation in Fig. 1) superimposed on the black smooth line. (Note: The (17) test is based on velocity changes at the GPS antenna which, depending on the antenna mount location above the vehicle rotation axis, could be generated by vehicle rolling transients as well as linear velocity changes of the vehicle.) As a result, the 3 deg γ_{Lmt} line in Fig. 1 would be crossed earlier than

29.7 sec, the first time "from above" being at 3 sec, thereby ending Coarse Leveling by the (17) test at 3 sec.

Trajectory B in Fig. 1 shows a case where a high initial horizontal acceleration rapidly generates a 50 fps horizontal velocity change, but only in a transitory sense, with the vehicle returning to its initial velocity after the transient decays (occurring, for example, for an aircraft passing through the slipstream of another larger aircraft, or passing in and out of a severe wind-shear). In this case, the 50 fps transient crosses the 3 deg limit line "from above" at 11 sec, causing the (17) test to end Coarse Leveling at 11 sec for 3 deg leveling accuracy, significantly less than the 29.7 sec figure shown in Table 1 for a 50 fps velocity change.

Finally, consider the black-lined trajectory C in Fig. 1 representing, for example, a ship under constant magnetic heading and throttle setting, moving into a region of 10 fps changed ocean current. Under calm sea-state conditions, the ship would gradually be accelerated by the changed current through a 10 fps velocity change in 30 sec. Table 1 shows that to achieve 1 deg leveling accuracy, 17.8 sec is required under a 10 fps velocity change. But this is based on a 10 fps velocity trajectory crossing the 1 deg γ_{Lmt} line at 17.8 sec. In Fig.1, however, the black-lined trajectory is below the 1 deg line at time zero, hence, would complete Coarse Leveling at the minimum allowable Coarse leveling time (e.g., 1.5 sec as in the previous section), and with a leveling error of less than 1 deg. Under high sea-state conditions (added gray oscillations to trajectory C), the initial increased velocity change oscillation shown in Fig. 1 would extend the leveling time to 2.5 seconds when the total velocity change crosses the 1 deg γ_{Lmt} line "from above".

From the previous discussion it should be apparent that the leveling times shown in Table 1 are more representative of worst case conditions, and that actual leveling times would likely be much less. Leveling times would generally be faster than the Table 1 values under short term dynamic velocity changes induced by high sea-sate conditions (for a ship) or severe air turbulence (for an aircraft). Finally, as mentioned previously, trajectory conditions for short leveling times can also be actively set by controlling the vehicle (manually or automatically), to maintain an approximate steady GPS based horizontal velocity, thus preventing the large Fig. 1 trajectory change conditions from ever occurring.

SUMMARY

Implementing the Coarse Leveling concept in the INS computer entails executing equations (5), (6), (9) - (11), (13), (17) - (19), and (23) - (25) summarized below and renumbered in order of execution.

$$C_B^{B_0} = I + \int_0^t \dot{C}_B^{B_0}\, dt \qquad \dot{C}_B^{B_0} = C_B^{B_0}\left(\underline{\omega}^B \times\right) \tag{29}$$

$$\underline{\upsilon}_{INS}^{B_0} = \int_0^t C_B^{B_0}\, \underline{a}_{SF_{INS}}^B\, dt \tag{30}$$

$$\underline{\upsilon}_{GPS}^{B} = \left(C_{B}^{B_0} \right)^{T} \underline{\upsilon}_{INS}^{B_0} - \left[\underline{\omega}^{B} - \left(C_{B}^{B_0} \right)^{T} \underline{\omega}_{0}^{B_0} \right] \times \underline{r}_{INS,GPS}^{B} \tag{31}$$

$$\underline{u}_{\upsilon GPS}^{B} = \underline{\upsilon}_{GPS}^{B} / \left| \underline{\upsilon}_{GPS}^{B} \right| \tag{32}$$

$$\Delta \underline{v}_{GPS,Ecntr}^{N} = \underline{v}_{GPS,Ecntr}^{N} - \underline{v}_{GPS,Ecntr_0}^{N} \tag{33}$$

$$\underline{\upsilon}_{GPS}^{N} = \Delta \underline{v}_{GPS,Ecntr_H}^{N} + \left(\Delta v_{GPS,Ecntr_{Up}} + g\, t \right) \underline{u}_{Up}^{N} \tag{34}$$

If t is greater than the minimum allowable time for Course Leveling, and $\left| \Delta \underline{v}_{GPS,Ecntr_H}^{N} \right| < \left| \underline{\upsilon}_{GPS}^{N} \right| \gamma_{Lmt}$, then complete Coarse Leveling: (35)

$$\underline{u}_{ZN*}^{B} = \underline{u}_{\upsilon GPS}^{B} \tag{36}$$

$$\begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} = \underline{u}_{ZN*}^{B} \tag{37}$$

$$\underline{u}_{YN*}^{B} = \begin{bmatrix} 0 \\ -C_{33} / \sqrt{C_{32}^{2} + C_{33}^{2}} \\ C_{32} / \sqrt{C_{32}^{2} + C_{33}^{2}} \end{bmatrix} \tag{38}$$

$$\underline{u}_{XN*}^{B} = \underline{u}_{YN*}^{B} \times \underline{u}_{ZN*}^{B} \tag{39}$$

$$C_{B}^{N*} = \left[\underline{u}_{XN*}^{B} \;\; \underline{u}_{YN*}^{B} \;\; \underline{u}_{ZN*}^{B} \right]^{T} \tag{40}$$

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