DIFFERENTIAL KINEMATICS OF POINT-TO-POINT RELATIVITY

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ABSTRACT

This article specializes and expands the kinematics of Point-To-Point Relativity to describe the differential motion of a remote point as observed at two spatial points in motion relative to one another. In its original form, Point-To-Point Relativity was restricted to constant relative velocity between the two observers (as in traditional Special Relativity). The differential approach described in this article places no restriction on relative velocity between observers. As a result, the equations for observed remote point acceleration also account for relative acceleration between the two observation points.

INTRODUCTION

Point-To-Point Relativity [1] is a revised form of traditional Relativity theory in which position is described as the distance vector between two points in space as viewed by observers translating relative to one-another. Unlike traditional Relativity theory, the Point-to-Point approach avoids the use of relatively translating reference frames, space-time diagrams, world lines intersecting with space-time events, and the concept of space-time simultaneity. In the Point-to-Point approach, distance vectors are represented as free vectors having no preferred location in reference frames in which they are described. An advantage for the new approach is eliminating the requirement for clock synchronization between the observers [2 Chpt. 8, 3 Sect. 12-2, 4 Chpt. VI Sect. 1]. As part of the Point-to-Point formulation, a new notation was developed in [1] to explicitly identify point-to-point distance-vectors/time-intervals measured by a particular observer, and their relationship with equivalent measurements taken by another observer.

This article describes a differential version of Point-to-Point kinematic, deriving Point-to-Point Lorentz formulas for the differential position change, velocity, and acceleration of a remote spatial point as determined by observers travelling relative to one-another. The original Point-To-Point article [1] was based on constant relative velocity between observers (as in traditional Special Relativity theory), initially deriving finite position change formulas as the basis for subsequent velocity and acceleration measurements. This article expands the scope of [1] to accommodate variations in relative velocity (acceleration) between the observation points.

The article first derives fundamental equations of differential kinematic position change between points in space for compatibility with either Newtonian or Relativistic geometries. The equations are then specialized for compatibility with the speed of light constancy requirement of Relativity theory. Deriving the position change equations in differential form allows them to be developed without assuming constant relative velocity between observation points (as in classical Special Relativity). The resulting Point-To-Point Relativity differential position change formulas are then used to develop the corresponding velocity and acceleration of a remote spatial point as determined by two observers in general motion (velocity and acceleration) relative to one another, to derive the differential equivalents of Lorentz "time dilation" and "length contraction", and to demonstrate the invariance between observers of classical Relativity "proper time".

NOTATION

The following general notation is used throughout the article:

$\underline{(\ )}$ = Vector parameter having length and direction.

$/\,i$ = Vector subscript denoting the vector parameter being observed (measured or calculated from measurements) at observation point i (i being point a or b).

Observable Event – An event at a position location in space at an instant in time (e.g., a lightning strike, explosion, or radar pulse illumination) that can be observed at a remote spatial location based on electro-magnetic wave propagation (e.g. light or radar) from the event to the observation point [2 pp. 29 & 36, 3 pp. 515 & 521, 4 pp. 28 & 236-238, 5 pp. 10].

GENERAL KINEMATIC EQUATIONS

Consider observers at points a and b observing the motion of a distant point p, each observer measuring the motion as the difference between observed p position locations ("events") at two successive time points t_1 and t_2 (t_2 following t_1):

$$\Delta\underline{x}_{p/a} = \underline{x}_{p_2/a} - \underline{x}_{p_1/a} \qquad \Delta\underline{x}_{p/b} = \underline{x}_{p_2/b} - \underline{x}_{p_1/b} \tag{1}$$

where $\underline{x}_{p_1/a}$, $\underline{x}_{p_2/a}$ are distance vectors (position) measured at point a from point a to p at times t_1 and t_2, $\Delta\underline{x}_{p/a}$ is the change (linear translation) in the point a observed p position vector over the t_1 to t_2 time interval, and similarly for $\underline{x}_{p_1/b}$, $\underline{x}_{p_2/b}$, $\Delta\underline{x}_{p/b}$.

If points a and b have translated during the t_1 to t_2 time interval, $\Delta \underline{x}_{p/a}$ will differ from $\Delta \underline{x}_{p/b}$. Observers a and b can account for the relative translation when predicting what the other would observe:

$$\Delta \underline{x}_{bp/a} = \Delta \underline{x}_{p/a} - \Delta \underline{x}_{b/a} \qquad \Delta \underline{x}_{ap/b} = \Delta \underline{x}_{p/b} - \Delta \underline{x}_{a/b} \tag{2}$$

where $\Delta \underline{x}_{bp/a}$ is the point a prediction of $\Delta \underline{x}_{p/b}$ and similarly for $\Delta \underline{x}_{ap/b}$. As in (1) we can also write

$$\Delta \underline{x}_{b/a} = \underline{x}_{b2/a} - \underline{x}_{b1/a} \qquad \Delta \underline{x}_{a/b} = \underline{x}_{a2/b} - \underline{x}_{a1/b} \tag{3}$$

Now assume that the t_1 to t_2 time interval is infinitesimally small. Then (2) becomes the differential equivalent:

$$d\underline{x}_{bp/a} = d\underline{x}_{p/a} - d\underline{x}_{b/a} \qquad d\underline{x}_{ap/b} = d\underline{x}_{p/b} - d\underline{x}_{a/b} \tag{4}$$

where $d(\,)$ is the differential equivalent of $\Delta(\,)$ over the infinitesimal time interval from t_1 to t_2. Based on Newtonian and Relativity theory symmetry and the principle of non-uniqueness between points a and b [6 pp. 423, 2 Chpt. 5, 7 pp. 177], we can further assume that $d\underline{x}_{b/a}$ and $d\underline{x}_{a/b}$ will be parallel but oppositely directed so that

$$\underline{v}_{b/a} \equiv \frac{d\underline{x}_{b/a}}{dt_a} = v_{b/a}\,\underline{u}_v \qquad \underline{v}_{a/b} \equiv \frac{d\underline{x}_{a/b}}{dt_b} = -\,v_{a/b}\,\underline{u}_v \qquad \underline{u}_v = \underline{v}_{b/a} / v_{b/a} = -\,\underline{v}_{a/b} / v_{a/b} \tag{5}$$

where $\underline{v}_{b/a}$ is the velocity of point b measured at point a, $v_{b/a}$ is the magnitude of $\underline{v}_{b/a}$, dt_a is an infinitesimal time interval for the $d\underline{x}_{b/a}$ position change as measured on a clock located at point a, similarly for $\underline{v}_{a/b}$ and dt_b, and $\underline{u}_v$ is a unit vector in the direction of $\underline{v}_{b/a}$ (and $d\underline{x}_{b/a}$).

Traditional Relativity and Newtonian theory invokes the principle that there is no “preferred” velocity reference location [6 pp. 423, 2 Chpt. 5, 7 pp. 177], hence, the velocity magnitude of point a observed at point b will equal the velocity magnitude of point b observed at point a:

$$v_{a/b} = v_{b/a} \equiv v_{ab} \tag{6}$$

where v_{ab} is the magnitude of the relative velocity between points a and b. From (5) and (6),

$$d\underline{x}_{b/a} = v_{ab}\,dt_a\,\underline{u}_v \qquad d\underline{x}_{a/b} = -\,v_{ab}\,dt_b\,\underline{u}_v \tag{7}$$

With (7), (4) becomes the equivalent differential of the classical Newtonian form [2 pp. 37, 3 pp. 508, 4 pp. 237, 5 pp.19]:

$$d\underline{x}_{bp/a} = d\underline{x}_{p/a} - v_{ab}\, dt_a\, \underline{u}_v \qquad d\underline{x}_{ap/b} = d\underline{x}_{p/b} + v_{ab}\, dt_b\, \underline{u}_v \tag{8}$$

For subsequent use, we also expand $d\underline{x}_{bp/a}$ and $d\underline{x}_{ap/b}$ in (8) into components parallel and perpendicular to the *b*-relative-to-*a* velocity vector direction $\underline{u}_v$. Recognizing that vector components along $\underline{u}_v$ are their dot products with $\underline{u}_v$:

$$d\underline{x}_{p/a} = d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v + d\underline{x}_{p/a_\perp} \qquad d\underline{x}_{p/b} = d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v + d\underline{x}_{p/b_\perp} \tag{9}$$

where subscript $\perp$ identifies a parameter's component perpendicular to $\underline{u}_v$.

RELATIVITY AND NEWTONIAN GEOMETRY COMPATIBILITY

Traditional Newtonian geometry would have $d\underline{x}_{ap/b} = d\underline{x}_{p/a}$, $d\underline{x}_{bp/a} = d\underline{x}_{p/b}$, and $dt_b = dt_a$ in (8). According to Relativity theory, however, these traditional equalities do not exactly hold along $\underline{u}_v$. To accommodate both, the (9) equalities are modified as in [4 pp. 236]:

$$d\underline{x}_{bp/a} = \alpha\, d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v + d\underline{x}_{p/b_\perp} \qquad d\underline{x}_{ap/b} = \alpha\, d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v + d\underline{x}_{p/a_\perp} \tag{10}$$

where α is set to 1 for Newtonian compatibility and to a different value (to be determined subsequently) for Relativity compatibility. Substituting (8) with (9) in (10) obtains after rearrangement (similar to [4 pp. 236]):

$$\begin{aligned}
\alpha\, d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v + d\underline{x}_{p/a_\perp} &= d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v + d\underline{x}_{p/b_\perp} + v_{ab}\, dt_b\, \underline{u}_v \\
\alpha\, d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v + d\underline{x}_{p/b_\perp} &= d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v + d\underline{x}_{p/a_\perp} - v_{ab}\, dt_a\, \underline{u}_v
\end{aligned} \tag{11}$$

The $\underline{u}_v$ components in (11) can be rearranged into

$$\begin{aligned}
d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v &= \frac{1}{\alpha}\left(d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v + v_{ab}\, dt_b\, \underline{u}_v \right) \\
&= d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v + v_{ab}\, dt_b\, \underline{u}_v + \left(\frac{1}{\alpha} - 1\right)\left(d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v + v_{ab}\, dt_b\, \underline{u}_v \right) \\
d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v &= \frac{1}{\alpha}\left(d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v - v_{ab}\, dt_a\, \underline{u}_v \right) \\
&= d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v - v_{ab}\, dt_a\, \underline{u}_v + \left(\frac{1}{\alpha} - 1\right)\left(d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v - v_{ab}\, dt_a\, \underline{u}_v \right)
\end{aligned} \tag{12}$$

Substituting $d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v$ from (12) into the (9) $d\underline{x}_{p/a}$ expression yields:

$$d\underline{x}_{p/a} = d\underline{x}_{p/b}\cdot\underline{u}_v\,\underline{u}_v + d\underline{x}_{p/a_\perp} + v_{ab}\,dt_b\,\underline{u}_v + \left(\frac{1}{\alpha}-1\right)\left(d\underline{x}_{p/b}\cdot\underline{u}_v\,\underline{u}_v + v_{ab}\,dt_b\,\underline{u}_v\right) \quad (13)$$

Then, recognizing from the perpendicular components in (11) that $d\underline{x}_{p/a_\perp} = d\underline{x}_{p/b_\perp}$, and from (9) that $d\underline{x}_{p/b}\cdot\underline{u}_v\,\underline{u}_v + d\underline{x}_{p/b_\perp} = d\underline{x}_{p/b}$, (13) becomes

$$d\underline{x}_{p/a} = d\underline{x}_{p/b} + v_{ab}\,dt_b\,\underline{u}_v + \left(\frac{1}{\alpha}-1\right)\left(d\underline{x}_{p/b}\cdot\underline{u}_v\,\underline{u}_v + v_{ab}\,dt_b\,\underline{u}_v\right) \qquad (14)$$

Using the identical procedure for the $d\underline{x}_{p/b}\cdot\underline{u}_v\,\underline{u}_v$ expression in (12) finds similarly for $d\underline{x}_{p/b}$:

$$d\underline{x}_{p/b} = d\underline{x}_{p/a} - v_{ab}\,dt_a\,\underline{u}_v + \left(\frac{1}{\alpha}-1\right)\left(d\underline{x}_{p/a}\cdot\underline{u}_v\,\underline{u}_v - v_{ab}\,dt_a\,\underline{u}_v\right) \qquad (15)$$

Eqs. (14) and (15) comprise a set of generalized differential distance vector conversion formulas (from observer *b* to *a* and from observer *a* to *b*) that are compatible with either Newtonian geometry or Relativity kinematic theory. For a complete conversion set it remains to find generalized equations for converting dt_b to dt_a and dt_a to dt_b.

Eqs. (14) and (15) can be inverted to find general solutions for the dt_a and dt_b differential time intervals. Taking the dot product of (14) with $\underline{u}_v$ obtains with rearrangement:

$$d\underline{x}_{p/a}\cdot\underline{u}_v = \frac{1}{\alpha}\left(d\underline{x}_{p/b}\cdot\underline{u}_v + v_{ab}\,dt_b\right) \qquad (16)$$

Substituting $d\underline{x}_{p/a}\cdot\underline{u}_v$ from (16) into (15) (multiplied by α dotted with $\underline{u}_v$) and solving for dt_a gives:

$$dt_a = \frac{1}{v_{b/a}}\left[\frac{1}{\alpha}\left(d\underline{x}_{p/b}\cdot\underline{u}_v + v_{ab}\,dt_b\right) - \alpha\,d\underline{x}_{p/b}\cdot\underline{u}_v\right]$$
$$= \frac{1}{\alpha\,v_{b/a}}\left(d\underline{x}_{p/b}\cdot\underline{u}_v + v_{ab}\,dt_b - \alpha^2 d\underline{x}_{p/b}\cdot\underline{u}_v\right) = \frac{1}{\alpha}\left[dt_b + \left(1-\alpha^2\right)d\underline{x}_{p/b}\cdot\underline{u}_v / v_{ab}\right] \qquad (17)$$

Similarly, dotting (15) with $\underline{u}_v$ and substituting the $d\underline{x}_{p/a}\cdot\underline{u}_v$ result into (14) (multiplied by α dotted with $\underline{u}_v$) solves for dt_b:

$$dt_b = \frac{1}{\alpha}\left[dt_a - \left(1-\alpha^2\right)d\underline{x}_{p/a}\cdot\underline{u}_v / v_{ab}\right] \qquad (18)$$

Eqs. (14), (15), (17), and (18) summarize as follows

$$
\begin{aligned}
d\underline{x}_{p/a} &= d\underline{x}_{p/b} + v_{ab}\, dt_b\, \underline{u}_v + \left(\frac{1}{\alpha} - 1\right)\left(d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v + v_{ab}\, dt_b\, \underline{u}_v\right) \\
dt_a &= \frac{1}{\alpha}\left[dt_b + \left(1 - \alpha^2\right) d\underline{x}_{p/b} \cdot \underline{u}_v / v_{ab}\right] \\
d\underline{x}_{p/b} &= d\underline{x}_{p/a} - v_{ab}\, dt_a\, \underline{u}_v + \left(\frac{1}{\alpha} - 1\right)\left(d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v - v_{ab}\, dt_a\, \underline{u}_v\right) \\
dt_b &= \frac{1}{\alpha}\left[dt_a - \left(1 - \alpha^2\right) d\underline{x}_{p/a} \cdot \underline{u}_v / v_{ab}\right]
\end{aligned}
\tag{19}
$$

Substituting $\underline{u}_v = \underline{v}_{b/a} / v_{ab}$ from (5) with (6), (19) becomes

$$
\begin{aligned}
d\underline{x}_{p/a} &= d\underline{x}_{p/b} + \underline{v}_{b/a}\, dt_b + \left(\frac{1}{\alpha} - 1\right)\left(d\underline{x}_{p/b} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 + \underline{v}_{b/a}\, dt_b\right) \\
dt_a &= \frac{1}{\alpha}\left[dt_b + \left(1 - \alpha^2\right) d\underline{x}_{p/b} \cdot \underline{v}_{b/a} / v_{ab}^2\right] \\
d\underline{x}_{p/b} &= d\underline{x}_{p/a} - \underline{v}_{b/a}\, dt_a + \left(\frac{1}{\alpha} - 1\right)\left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a\right) \\
dt_b &= \frac{1}{\alpha}\left[dt_a - \left(1 - \alpha^2\right) d\underline{x}_{p/a} \cdot \underline{v}_{b/a} / v_{ab}^2\right]
\end{aligned}
\tag{20}
$$

An equivalent symmetrical version of (20) derives from (5) using $\underline{u}_v = -\,\underline{v}_{a/b} / v_{ab}$ for the first equation set in (19) and $\underline{u}_v = \underline{v}_{b/a} / v_{ab}$ for the second equation set in (19):

$$
\begin{aligned}
d\underline{x}_{p/a} &= d\underline{x}_{p/b} - \underline{v}_{a/b}\, dt_b + \left(\frac{1}{\alpha} - 1\right)\left(d\underline{x}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2 - \underline{v}_{a/b}\, dt_b\right) \\
dt_a &= \frac{1}{\alpha}\left[dt_b - \left(1 - \alpha^2\right) d\underline{x}_{p/b} \cdot \underline{v}_{a/b} / v_{ab}^2\right] \\
d\underline{x}_{p/b} &= d\underline{x}_{p/a} - \underline{v}_{b/a}\, dt_a + \left(\frac{1}{\alpha} - 1\right)\left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a\right) \\
dt_b &= \frac{1}{\alpha}\left[dt_a - \left(1 - \alpha^2\right) d\underline{x}_{p/a} \cdot \underline{v}_{b/a} / v_{ab}^2\right]
\end{aligned}
\tag{21}
$$

Eqs. (19), (20), or (21), with (6) constitute generalized sets of Point-to-Point kinematic conversion formulas that are compatible with either Newtonian geometry or Relativity theory. The distinguishing characteristic between either is the value selected for α. For $\alpha = 1$, Eqs. (19) – (21) reduce to the classic Newtonian form. For compatibility with Relativity theory, α must be set so that the speed-of-light constancy law of Relativity theory is satisfied.

SETTING ALPHA FOR RELATIVITY COMPATIBILITY

For compatibility with Relativity theory, α in (19) – (21) is used to account for experimental and theoretical findings that the speed of light (or any electro-magnetic wave speed) is the same constant to any observer in the same isotropic homogenous medium (or in a vacuum) [8, 9 Sect. II]. Thus, consider what observers a and b would measure for the distance a photon r of light would travel between two observable points ("events") in space (events 1 and 2, event 2 following event 1). Each observer would find:

$$\left|\Delta \underline{x}_{r/a}\right|^2 = \Delta \underline{x}_{r/a} \cdot \Delta \underline{x}_{r/a} = c^2\, \Delta t_a^2 \qquad \left|\Delta \underline{x}_{r/b}\right|^2 = \Delta \underline{x}_{r/b} \cdot \Delta \underline{x}_{r/b} = c^2\, \Delta t_b^2 \qquad (22)$$

where $\Delta \underline{x}_{r/a}$ is the distance vector traversed by photon r between event points 1 and 2, Δt_a is the time interval recorded on a point a located clock between observations 1 and 2, c is the speed of light, and similarly for $\Delta \underline{x}_{r/b}$, Δt_b. The equivalent of (22) over an infinitesimal space time interval would be

$$\left|d \underline{x}_{r/a}\right|^2 = d \underline{x}_{r/a} \cdot d \underline{x}_{r/a} = c^2\, dt_a^2 \qquad \left|d \underline{x}_{r/b}\right|^2 = d \underline{x}_{r/b} \cdot d \underline{x}_{r/b} = c^2\, dt_b^2 \qquad (23)$$

Applying the generalized previous results to photon motion equates point p to photon r for which (9), (23), and Pythagoras give

$$\begin{aligned} \left|d \underline{x}_{r/a}\right|^2 &= \left(d \underline{x}_{r/a} \cdot \underline{u}_v\right)^2 + d \underline{x}_{r/a_\perp} \cdot d \underline{x}_{r/a_\perp} = c^2\, dt_a^2 \\ \left|d \underline{x}_{r/b}\right|^2 &= \left(d \underline{x}_{r/b} \cdot \underline{u}_v\right)^2 + d \underline{x}_{r/b_\perp} \cdot d \underline{x}_{r/b_\perp} = c^2\, dt_b^2 \end{aligned} \qquad (24)$$

Taking the difference between the (24) expressions finds

$$\begin{aligned} &\left(d \underline{x}_{r/b} \cdot \underline{u}_v\right)^2 - \left(d \underline{x}_{r/a} \cdot \underline{u}_v\right)^2 \\ &= c^2 \left(dt_b^2 - dt_a^2\right) + d \underline{x}_{p/a_\perp} \cdot d \underline{x}_{p/a_\perp} - d \underline{x}_{p/b_\perp} \cdot d \underline{x}_{p/b_\perp} \end{aligned} \qquad (25)$$

For this exercise it is convenient to use the equivalent (11) form of the (19) position change conversion pair. Identifying photon r as p in (11) obtains

$$\begin{aligned} \alpha\, d \underline{x}_{r/a} \cdot \underline{u}_v\, \underline{u}_v + d \underline{x}_{r/a_\perp} &= d \underline{x}_{r/b} \cdot \underline{u}_v\, \underline{u}_v + d \underline{x}_{r/b_\perp} + v_{ab}\, dt_b\, \underline{u}_v \\ \alpha\, d \underline{x}_{r/b} \cdot \underline{u}_v\, \underline{u}_v + d \underline{x}_{r/b_\perp} &= d \underline{x}_{r/a} \cdot \underline{u}_v\, \underline{u}_v + d \underline{x}_{r/a_\perp} - v_{ab}\, dt_a\, \underline{u}_v \end{aligned} \qquad (26)$$

The components of (26) parallel and perpendicular to $\underline{u}_v$ can be written individually as

$$\begin{aligned} \alpha\, d\underline{x}_{r/a} \cdot \underline{u}_v &= d\underline{x}_{r/b} \cdot \underline{u}_v + v_{ab}\, dt_b \\ \alpha\, d\underline{x}_{r/b} \cdot \underline{u}_v &= d\underline{x}_{r/a} \cdot \underline{u}_v - v_{ab}\, dt_a \\ d\underline{x}_{r/b\perp} &= d\underline{x}_{r/a\perp} \end{aligned} \tag{27}$$

Solving for $d\underline{x}_{r/b} \cdot \underline{u}_v$ from the first expression in (27) and substitution in the second obtains with rearrangement:

$$\left(\alpha^2 - 1\right) d\underline{x}_{r/a} \cdot \underline{u}_v = v_{ab}\left(\alpha\, dt_b - dt_a\right) \tag{28}$$

Similarly, solving for $d\underline{x}_{r/a} \cdot \underline{u}_v$ from the second expression in (27) and substitution in the first obtains:

$$\left(\alpha^2 - 1\right) d\underline{x}_{r/b} \cdot \underline{u}_v = v_{ab}\left(dt_b - \alpha\, dt_a\right) \tag{29}$$

Squaring (28) and (29), and taking their difference gives

$$\begin{aligned} \left(1-\alpha^2\right)^2 \left[\left(d\underline{x}_{r/b} \cdot \underline{u}_v\right)^2 - \left(d\underline{x}_{r/a} \cdot \underline{u}_v\right)^2\right] &= v_{ab}^2 \left[\left(dt_b - \alpha\, dt_a\right)^2 - \left(\alpha\, dt_b - dt_a\right)^2\right] \\ &= v_{ab}^2 \left[dt_b^2 + \left(\alpha\, dt_a\right)^2 - \left(\alpha\, dt_b\right)^2 - dt_a^2\right] = v_{ab}^2 \left(1-\alpha^2\right)\left(dt_b^2 - dt_a^2\right) \end{aligned} \tag{30}$$

or

$$\left(1-\alpha^2\right)\left[\left(d\underline{x}_{r/b} \cdot \underline{u}_v\right)^2 - \left(d\underline{x}_{r/a} \cdot \underline{u}_v\right)^2\right] = v_{ab}^2 \left(dt_b^2 - dt_a^2\right) \tag{31}$$

Substituting (25) in (31) and applying the third expression from (27) finds

$$\left(1-\alpha^2\right) c^2 \left(dt_b^2 - dt_a^2\right) = v_{ab}^2 \left(dt_b^2 - dt_a^2\right) \tag{32}$$

or

$$1 - \alpha^2 = v_{ab}^2 / c^2 \tag{33}$$

Eq. (33) easily solves for α, yielding the well-known Relativity coefficient:

$$\alpha = \sqrt{1 - v_{ab}^2 / c^2} \tag{34}$$

POINT-TO-POINT RELATIVITY OBSERVATIONS OF REMOTE POINT DIFFERENTIAL POSITION CHANGE

With (33) and (34), Eqs. (21) become Differential Point-To-Point Relativity conversion formulas relating remote point p differential position change as viewed by observers at points a and b:

$$d\underline{x}_{p/a} = d\underline{x}_{p/b} - \underline{v}_{a/b}\, dt_b + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2 - \underline{v}_{a/b}\, dt_b \right)$$

$$dt_a = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_b - d\underline{x}_{p/b} \cdot \underline{v}_{a/b} / c^2 \right) \qquad (35)$$

$$d\underline{x}_{p/b} = d\underline{x}_{p/a} - \underline{v}_{b/a}\, dt_a + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a \right)$$

$$dt_b = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_a - d\underline{x}_{p/a} \cdot \underline{v}_{b/a} / c^2 \right)$$

Eqs. (35) are the differential Point-To-Point Relativity conversion equivalent of the general Lorentz transformation operations in traditional Relativity theory [3 Eqs. (12-5a); 5 Eqs. (10.32) - (10.33), (10-36) - (10.37) & pp. 30]. Note that the $\underline{v}_{a/b}$, $\underline{v}_{b/a}$, v_{ab} relative velocity terms between points a and b as defined by (5) with (6) are completely general without any constancy assumption. This characteristic will carry forward into the subsequent derivation of p acceleration viewed from points a and b in which relative acceleration between a and b is included.

POINT-TO-POINT RELATIVITY OBSERVATIONS OF REMOTE POINT VELOCITY

The derivation of remote point p velocity relative to observation point a begins with a restatement of the (35) observation equations for observer a:

$$d\underline{x}_{p/a} = d\underline{x}_{p/b} - \underline{v}_{a/b}\, dt_b + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2 - \underline{v}_{a/b}\, dt_b \right) \qquad (36)$$

$$dt_a = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_b - d\underline{x}_{p/b} \cdot \underline{v}_{a/b} / c^2 \right)$$

Dividing (36) by dt_b finds with rearrangement

$$\frac{d\underline{x}_{p/a}}{dt_a}\frac{dt_a}{dt_b}=\frac{d\underline{x}_{p/b}}{dt_b}-\frac{d\underline{x}_{p/b}}{dt_b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2+\frac{1}{\sqrt{1-v_{a/b}^2/c^2}}\left(\frac{d\underline{x}_{p/b}}{dt_b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2-\underline{v}_{a/b}\right)$$

$$\frac{dt_a}{dt_b}=\frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(1-\frac{d\underline{x}_{p/b}}{dt_b}\cdot\underline{v}_{a/b}/c^2\right) \tag{37}$$

Defining

$$\underline{v}_{p/a}\equiv\frac{d\underline{x}_{p/a}}{dt_a}\qquad \underline{v}_{p/b}\equiv\frac{d\underline{x}_{p/b}}{dt_b} \tag{38}$$

where $\underline{v}_{p/a}$ and $\underline{v}_{p/b}$ are the point *p* velocities observed at points *a* and *b*. With (38), (37) becomes

$$\underline{v}_{p/a}\frac{dt_a}{dt_b}=\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2+\frac{1}{\sqrt{1-v_{a/b}^2/c^2}}\left(\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2-\underline{v}_{a/b}\right)$$

$$\frac{dt_a}{dt_b}=\frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right) \tag{39}$$

Substituting the second expression in (39) into the first obtains

$$\underline{v}_{p/a}\frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)$$

$$=\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{a/b}^2+\frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2-\underline{v}_{a/b}\right) \tag{40}$$

Rearrangement of (40) then yields the Point-to-Point Relativity equation for $\underline{v}_{p/a}$, point *p* velocity relative to point *a*, as a function of $\underline{v}_{p/b}$, point *p* velocity relative to point *b*:

$$\underline{v}_{p/a}=\frac{\sqrt{1-v_{ab}^2/c^2}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)}-\frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right)\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)} \tag{41}$$

Substituting $\underline{v}_{a/b}=-\underline{v}_{b/a}$ from (5) – (6) into (41) would obtain the equivalent to what has previously been found by traditional Relativity theory [3 pp. Eq. (12-12), 5 pp. Eq. (16.07)].

The velocity of point *p* velocity relative to point *b* is found similarly, but starting with the (35) observation equations for observer *b*. The result is

$$\underline{v}_{p/b} = \frac{\sqrt{1 - v_{ab}^2 / c^2}\left(\underline{v}_{p/a} - \underline{v}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2\right)}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a} / c^2\right)} - \frac{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a} / v_{ab}^2\right) \underline{v}_{b/a}}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a} / c^2\right)} \qquad (42)$$

POINT-TO-POINT RELATIVITY OBSERVATIONS OF REMOTE POINT ACCELERATION

Deriving point p acceleration relative to observation point a begins with the (41) equation for $\underline{v}_{p/a}$, the dt_a / dt_b equation in (39), and the acceleration definitions in (36). The derivation will also assume that the original (5) – (6) premise of $\underline{v}_{a/b} = -\underline{v}_{b/a}$ is valid under changing $\underline{v}_{b/a}$. Taking the differential of (41) with $d\underline{v}_{p/b}$ terms grouped first and $d\underline{v}_{b/a}$ terms second yields

$$\begin{aligned}
d\underline{v}_{p/a} = {} & \frac{\sqrt{1 - v_{ab}^2 / c^2}\left(d\underline{v}_{p/b} - d\underline{v}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2\right)}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right)} \\
& + \frac{\sqrt{1 - v_{ab}^2 / c^2}\left(\underline{v}_{p/b} - \underline{v}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2\right) d\underline{v}_{p/b} \cdot \underline{v}_{a/b}}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right)^2 c^2} + \frac{d\underline{v}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b}}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right) v_{ab}^2} \\
& - \frac{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / v_{ab}^2\right) d\underline{v}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b}}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right)^2 c^2} \\
& + \frac{dv_{ab}^2\left(\underline{v}_{p/b} - \underline{v}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2\right)}{2\sqrt{1 - v_{ab}^2 / c^2}\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right) c^2} \\
& - \frac{\sqrt{1 - v_{ab}^2 / c^2}\begin{pmatrix} \underline{v}_{p/b} \cdot d\underline{v}_{a/b}\, \underline{v}_{a/b} + \underline{v}_{p/b} \cdot \underline{v}_{a/b}\, d\underline{v}_{a/b} \\ - dv_{ab}^2\, \underline{v}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2 \end{pmatrix}}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right) v_{ab}^2} \\
& + \frac{\sqrt{1 - v_{ab}^2 / c^2}\left(\underline{v}_{p/b} - \underline{v}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2\right) \underline{v}_{p/b} \cdot d\underline{v}_{a/b} / c^2}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right)^2} \\
& - \frac{dv_{ab}^2\, \underline{v}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b}}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right) v_{ab}^4} - \frac{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / v_{ab}^2\right) d\underline{v}_{a/b}}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right)} \\
& + \frac{\underline{v}_{p/b} \cdot d\underline{v}_{a/b}\, \underline{v}_{a/b}}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right) v_{ab}^2} - \frac{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / v_{ab}^2\right) \underline{v}_{p/b} \cdot d\underline{v}_{a/b}\, \underline{v}_{a/b}}{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{a/b} / c^2\right)^2 c^2}
\end{aligned} \qquad (43)$$

The dv_{ab}^2 term in (43) is obtained from

$$v_{ab}^2 = \underline{v}_{b/a} \cdot \underline{v}_{b/a} = \underline{v}_{a/b} \cdot \underline{v}_{a/b} \quad \rightarrow \quad dv_{ab}^2 = 2\, \underline{v}_{b/a} \,.\, d\underline{v}_{b/a} = 2\, \underline{v}_{a/b} \,.\, d\underline{v}_{a/b} \qquad (44)$$

Substituting $dv_{ab}^2 = 2\, \underline{v}_{a/b} \,.\, d\underline{v}_{a/b}$ from (44) in (43) gives

$$\begin{aligned}
d\underline{v}_{p/a} = {} & \frac{\sqrt{1-v_{ab}^2/c^2}\left(d\underline{v}_{p/b} - d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)} \\
& + \frac{\sqrt{1-v_{ab}^2/c^2}\left(\underline{v}_{p/b} - \underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right) d\underline{v}_{p/b}\cdot\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 c^2} + \frac{d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right) v_{ab}^2} \\
& - \frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right) d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 c^2} \\
& + \frac{\underline{v}_{a/b}\,.\,d\underline{v}_{a/b}\left(\underline{v}_{p/b} - \underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\sqrt{1-v_{ab}^2/c^2}\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right) c^2} \\
& - \frac{\sqrt{1-v_{ab}^2/c^2}\begin{pmatrix} \underline{v}_{p/b}\cdot d\underline{v}_{a/b}\,\underline{v}_{a/b} + \underline{v}_{p/b}\cdot\underline{v}_{a/b}\,d\underline{v}_{a/b} \\ -2\,\underline{v}_{a/b}\,.\,d\underline{v}_{a/b}\,\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2 \end{pmatrix}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right) v_{ab}^2} \qquad (45) \\
& + \frac{\sqrt{1-v_{ab}^2/c^2}\,\underline{v}_{p/b}\,.\,d\underline{v}_{a/b}\left(\underline{v}_{p/b} - \underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 c^2} \\
& - \frac{2\,\underline{v}_{a/b}\,.\,d\underline{v}_{a/b}\,\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right) v_{ab}^4} - \frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right) d\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)} \\
& + \frac{\underline{v}_{p/b}\,.\,d\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right) v_{ab}^2} - \frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right)\underline{v}_{p/b}\,.\,d\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 c^2}
\end{aligned}$$

The third and fourth terms in (45) combine as

$$\frac{d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)v_{ab}^2}-\frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right)d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2c^2}$$

$$=\frac{d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2}\left[\frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)}{v_{a/b}^2}-\frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right)}{c^2}\right] \quad (46)$$

$$=\frac{d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2}\left(\frac{1}{v_{ab}^2}-\frac{1}{c^2}\right)=\frac{\left(1-v_{ab}^2/c^2\right)d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2v_{ab}^2}$$

The last two terms in (45) combine similarly:

$$\frac{\underline{v}_{p/b}\cdot d\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)v_{ab}^2}-\frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right)\underline{v}_{p/b}\cdot d\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2c^2}$$

$$=\frac{\left(1-v_{ab}^2/c^2\right)\underline{v}_{p/b}\cdot d\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2v_{ab}^2} \quad (47)$$

With (46) and (47), (45) becomes

$$\begin{aligned}d\underline{v}_{p/a}=&\frac{\left(1-v_{ab}^2/c^2\right)d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2v_{a/b}^2}+\frac{\sqrt{1-v_{ab}^2/c^2}\left(d\underline{v}_{p/b}-d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)}\\&+\frac{\sqrt{1-v_{ab}^2/c^2}\,d\underline{v}_{p/b}\cdot\underline{v}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2c^2}\\&+\frac{\underline{v}_{a/b}\cdot d\underline{v}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\sqrt{1-v_{ab}^2/c^2}\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)c^2}\\&-\frac{\sqrt{1-v_{ab}^2/c^2}\begin{pmatrix}\underline{v}_{p/b}\cdot d\underline{v}_{a/b}\,\underline{v}_{a/b}+\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,d\underline{v}_{a/b}\\-2\,\underline{v}_{a/b}\cdot d\underline{v}_{a/b}\,\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\end{pmatrix}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)v_{ab}^2}\end{aligned} \quad (48)$$

Continued

$$+\frac{\sqrt{1-v_{a/b}^2/c^2}\,\underline{v}_{p/b}\cdot d\underline{v}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 c^2}$$

$$-\frac{2\,\underline{v}_{a/b}\cdot d\underline{v}_{a/b}\,\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)v_{ab}^4}-\frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right)d\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)} \qquad \text{(48) Concluded}$$

$$+\frac{\left(1-v_{ab}^2/c^2\right)\underline{v}_{p/b}\cdot d\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 v_{ab}^2}$$

Define

$$\underline{a}_{p/a}\equiv\frac{d\underline{v}_{p/a}}{dt_a} \qquad \underline{a}_{p/b}\equiv\frac{d\underline{v}_{p/b}}{dt_b} \qquad \underline{a}_{b/a}\equiv\frac{d\underline{v}_{b/a}}{dt_a} \qquad \underline{a}_{a/b}\equiv\frac{d\underline{v}_{a/b}}{dt_b} \qquad (49)$$

where $\underline{a}_{p/a}$, $\underline{a}_{p/b}$ are point p accelerations observed at points *a* and *b*, $\underline{a}_{b/a}$ is point observation point *b* acceleration observed at point a, and $\underline{a}_{a/b}$ is observation point *a* acceleration observed at point *a*. Dividing (48) by dt_a and applying (49) then gives

$$
\underline{a}_{p/a} = \left[\begin{aligned}
&\frac{\left(1-v_{ab}^2/c^2\right)\underline{a}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 v_{ab}^2} + \frac{\sqrt{1-v_{ab}^2/c^2}\left(\underline{a}_{p/b}-\underline{a}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)} \\
&+\frac{\sqrt{1-v_{ab}^2/c^2}\,\underline{a}_{p/b}\cdot\underline{v}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 c^2} \\
&+\frac{\underline{v}_{a/b}\cdot\underline{a}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\sqrt{1-v_{ab}^2/c^2}\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)c^2} \\
&-\frac{\sqrt{1-v_{ab}^2/c^2}\begin{pmatrix}\underline{v}_{p/b}\cdot\underline{a}_{a/b}\,\underline{v}_{a/b}+\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{a}_{a/b}\\ -2\,\underline{v}_{a/b}\cdot\underline{a}_{a/b}\,\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\end{pmatrix}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)v_{ab}^2} \\
&+\frac{\sqrt{1-v_{ab}^2/c^2}\,\underline{v}_{p/b}\cdot\underline{a}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 c^2} \\
&-\frac{2\,\underline{v}_{a/b}\cdot\underline{a}_{a/b}\,\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)v_{ab}^4} - \frac{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right)\underline{a}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)} \\
&+\frac{\left(1-v_{ab}^2/c^2\right)\underline{v}_{p/b}\cdot\underline{a}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 v_{ab}^2}
\end{aligned} \right] \Bigg/ \left(\frac{dt_a}{dt_b}\right) \tag{50}
$$

Substituting dt_a/dt_b from (39) into (50) finally obtains

$$
\begin{aligned}
\underline{a}_{p/a} = & \frac{\left(1-v_{ab}^2/c^2\right)^{3/2} \underline{a}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^3 v_{ab}^2} + \frac{\left(1-v_{ab}^2/c^2\right)\left(\underline{a}_{p/b}-\underline{a}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2} \\
& + \frac{\left(1-v_{ab}^2/c^2\right)\underline{a}_{p/b}\cdot\underline{v}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^3 c^2} \\
& + \frac{\underline{v}_{a/b}\cdot\underline{a}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 c^2} \\
& - \frac{\left(1-v_{ab}^2/c^2\right)\begin{pmatrix}\underline{v}_{p/b}\cdot\underline{a}_{a/b}\,\underline{v}_{a/b}+\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{a}_{a/b} \\ -2\,\underline{v}_{a/b}\cdot\underline{a}_{a/b}\,\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\end{pmatrix}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 v_{ab}^2} \\
& + \frac{\left(1-v_{ab}^2/c^2\right)\underline{v}_{p/b}\cdot\underline{a}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^3 c^2} \\
& + \frac{\left(1-v_{ab}^2/c^2\right)^{3/2}\underline{v}_{p/b}\cdot\underline{a}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^3 v_{ab}^2} - \frac{\sqrt{1-v_{a/b}^2/c^2}\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/v_{ab}^2\right)\underline{a}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2} \\
& - \frac{2\sqrt{1-v_{ab}^2/c^2}\,\underline{v}_{a/b}\cdot\underline{a}_{a/b}\,\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2 v_{ab}^4}
\end{aligned}
\tag{51}
$$

Eq. (51) defines the acceleration of point *p* observed at point *a* as a function of *p* acceleration observed at point *b* and the relative velocity/acceleration between observation points *a* and *b*. It is also to be noted that under constant relative velocity between points *a* and *b* (i.e., constant $\underline{v}_{b/a}$ or $\underline{a}_{b/a}$ = 0, the traditional assumption in classical Relativity theory), (51) reduces to

$$
\begin{aligned}
\underline{a}_{p/a} = & \frac{\left(1-v_{ab}^2/c^2\right)^{3/2} \underline{a}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^3 v_{ab}^2} + \frac{\left(1-v_{ab}^2/c^2\right)\left(\underline{a}_{p/b}-\underline{a}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^2} \\
& + \frac{\left(1-v_{ab}^2/c^2\right)\underline{a}_{p/b}\cdot\underline{v}_{a/b}\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{v}_{a/b}\,\underline{v}_{a/b}/v_{ab}^2\right)}{\left(1-\underline{v}_{p/b}\cdot\underline{v}_{a/b}/c^2\right)^3 c^2}
\end{aligned}
\tag{52}
$$

Substituting $\underline{v}_{a/b} = -\,\underline{v}_{b/a}$ from (5) – (6), the (52) result is equivalent to what has been obtained previously based on traditional Special Relativity [1 Eq. (86) & 3 Eqs. (12-13)].

Results similar to (52) can be formulated for the acceleration of point *p* observed at point *b* as a function of *p* acceleration observed at point *a* and the relative velocity/acceleration between observation points *a* and *b*. Starting with the (42) equation for $\underline{v}_{p/b}$, the dt_b equation in (35), and the acceleration definitions in (49), the final result is

$$
\begin{aligned}
\underline{a}_{p/b} = {} & \frac{\left(1 - v_{ab}^2/c^2\right)^{3/2} \underline{a}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a}}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a}/c^2\right)^3 v_{ab}^2} + \frac{\left(1 - v_{ab}^2/c^2\right)\left(\underline{a}_{p/a} - \underline{a}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2\right)}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a}/c^2\right)^2} \\
& + \frac{\left(1 - v_{ab}^2/c^2\right) \underline{a}_{p/a} \cdot \underline{v}_{b/a} \left(\underline{v}_{p/a} - \underline{v}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2\right)}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a}/c^2\right)^3 c^2} \\
& + \frac{\underline{v}_{b/a} \cdot \underline{a}_{b/a} \left(\underline{v}_{p/a} - \underline{v}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2\right)}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a}/c^2\right)^2 c^2} \\
& - \frac{\left(1 - v_{ab}^2/c^2\right) \begin{pmatrix} \underline{v}_{p/a} \cdot \underline{a}_{b/a}\, \underline{v}_{b/a} + \underline{v}_{p/a} \cdot \underline{v}_{b/a}\, \underline{a}_{b/a} \\ -\,2\, \underline{v}_{b/a} \cdot \underline{a}_{b/a}\, \underline{v}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 \end{pmatrix}}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a}/c^2\right)^2 v_{ab}^2} \\
& + \frac{\left(1 - v_{ab}^2/c^2\right) \underline{v}_{p/a} \cdot \underline{a}_{b/a} \left(\underline{v}_{p/a} - \underline{v}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2\right)}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a}/c^2\right)^3 c^2} \\
& + \frac{\left(1 - v_{ab}^2/c^2\right)^{3/2} \underline{v}_{p/a} \cdot \underline{a}_{b/a}\, \underline{v}_{b/a}}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a}/c^2\right)^3 v_{ab}^2} - \frac{\sqrt{1 - v_{b/a}^2/c^2}\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a} / v_{ab}^2\right) \underline{a}_{b/a}}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a}/c^2\right)^2} \qquad (53) \\
& - \frac{2\sqrt{1 - v_{ab}^2/c^2}\; \underline{v}_{b/a} \cdot \underline{a}_{b/a}\, \underline{v}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a}}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{b/a}/c^2\right) v_{ab}^4}
\end{aligned}
$$

Note the symmetry between (51) and (53), the expected result consistent with the basic premise of Relativity theory, and that can also be used as a validity check on the (53) derivation process.

DIFFERENTIAL POINT-TO-POINT FORMULAS FOR RELATIVE OBSERVER MOTION

Previous developments have centered on point *a* compared to *b* observations of the kinematic motion of a remote point *p* (i.e., a three-point solution). In this section we formulate a two-point kinematic solution by superimposing points *b* and *p*. Then the motion of point *p* observed at point *b* will be zero, the motion of point *p* observed at point *a* will represent the relative motion between observation points *a* and *b*, and three-point solution Eqs. (35), (41), (42), (51), and (53) collapse at point *b* into the simplified forms:

$$\text{For Point } p \text{ Located At Observation Point } b:$$

$$\begin{gathered}
d\underline{x}_{p/a} = d\underline{x}_{b/a} = -\frac{1}{\sqrt{1-v_{ab}^2/c^2}}\,\underline{v}_{a/b}\,dt_b \qquad dt_a = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\,dt_b \\
\underline{v}_{p/a} = \underline{v}_{b/a} = -\,\underline{v}_{a/b} \qquad \underline{a}_{p/a} = \underline{a}_{b/a} = -\sqrt{1-v_{a/b}^2/c^2}\;\underline{a}_{a/b} \\
d\underline{x}_{p/b} = d\underline{x}_{b/b} = 0 \qquad dt_b = \sqrt{1-v_{ab}^2/c^2}\;dt_a \\
\underline{v}_{p/b} = \underline{v}_{b/b} = 0 \qquad \underline{a}_{p/b} = \underline{a}_{b/b} = 0
\end{gathered} \tag{54}$$

A two point solution can also be developed for observer *b* by assuming another point *q* to be at location *a*. Then Eqs. (35), (41), (42), (51), and (53) (with *p* defined as *q* = *a*) collapse at point *a* into the simplified forms:

$$\text{For Point } q \text{ Located At Observation Point } a:$$

$$\begin{gathered}
d\underline{x}_{q/b} = d\underline{x}_{a/b} = -\frac{1}{\sqrt{1-v_{ab}^2/c^2}}\,\underline{v}_{b/a}\,dt_a \qquad dt_b = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\,dt_a \\
\underline{v}_{q/b} = \underline{v}_{a/b} = -\,\underline{v}_{b/a} \qquad \underline{a}_{q/b} = \underline{a}_{a/b} = -\sqrt{1-v_{a/b}^2/c^2}\;\underline{a}_{b/a} \\
d\underline{x}_{q/a} = d\underline{x}_{a/a} = 0 \qquad dt_a = \sqrt{1-v_{ab}^2/c^2}\;dt_b \\
\underline{v}_{q/a} = \underline{v}_{a/a} = 0 \qquad \underline{a}_{q/a} = \underline{a}_{a/a} = 0
\end{gathered} \tag{55}$$

Note the symmetry between (54) and (55), consistent with the basic premise of Relativity theory.

DIFFERENTIAL POINT-TO-POINT RELATIVITY TIME DILATION, LENGTH CONTRACTION, AND PROPER TIME

Two well-known consequences of traditional Relativity theory are the lengthening of time intervals (time dilation) and shorting of distances (distance contraction) predicted by Lorentz analytics [2 Chpt. 12, 3 pp. 517, 4 pp. 250, 5 Sect. 14 & 15]. In traditional Relativity, Lorentz analytics also defines a combined distance/time “proper time” parameter that has the same value when evaluated in reference frames translating relative to one-another [3 pp. 519, 5 Sect. 12]. The same effects arise with Differential Point-to-Point Relativity.

Differential Point-To-Point Relativity Time Dilation

Differential Point-to-Point Lorentz time dilation has already been demonstrated within the two-point solutions where events occurring at one observer were seen by the other observer. When point p differential position change occurred at observation point b, results in (54) showed that $dt_a = dt_b / \sqrt{1 - v_{ab}^2 / c^2}$, i.e., the time interval measured at point a during the spatial movement of point b, was longer than the same time interval measured at point b. Similarly, when point q differential position change occurred at observation point a, results in (55) showed that $dt_b = dt_a / \sqrt{1 - v_{ab}^2 / c^2}$, i.e., the time interval measured at point b during the spatial movement of point q over the time, was longer than the same time interval measured at point a. The effect is known as Lorentz "time dilation". The results are equivalent to what has been obtained from traditional Relativity theory [2 Chpt. 12, 3 pp. 517, 4 pp. 248 - 250, 5 Sect. 15].

Differential Point-To-Point Length Contraction

Differential Point-To-Point Lorentz distance contraction can be analytically demonstrated in general from (35) for simultaneous observation of the beginning and end points of p at point b (i.e., $dt_b = 0$) which finds

$$dt_a = d\underline{x}_{p/a} \cdot \underline{v}_{b/a} / c^2 \tag{56}$$

Substituting (56) in the (35) $d\underline{x}_{p/b}$ expression then finds:

$$\begin{aligned}
d\underline{x}_{p/b} &= d\underline{x}_{p/a} - \underline{v}_{b/a}\, dt_a + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a \right) \\
&= d\underline{x}_{p/a} - d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 + \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a \right) \\
&= d\underline{x}_{p/a} - d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 \\
&\quad + \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, d\underline{x}_{p/a} \cdot \underline{v}_{b/a} / c^2 \right) \\
&= d\underline{x}_{p/a} - d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 + \frac{\left(1 - v_{ab}^2 / c^2\right)}{\sqrt{1 - v_{ab}^2 / c^2}}\, d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 \\
&= d\underline{x}_{p/a} - d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 + \sqrt{1 - v_{ab}^2 / c^2}\, d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 \\
&= d\underline{x}_{p/a} - \left(1 - \sqrt{1 - v_{ab}^2 / c^2}\right) d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2
\end{aligned} \tag{57}$$

Eq. (57) shows that $d\underline{x}_{p/b}$, the differential distance vector seen by observer *b* at a fixed time instant will equal the $d\underline{x}_{p/a}$ differential distance determined by observer *a*, but with the component parallel to $\underline{v}_{ab/a}$ shortened by the factor $1-\sqrt{1-v_{ab}^2 / c^2} \approx \frac{1}{2} v_{ab}^2 / c^2$. The effect is known as "Lorentz distance contraction". The result is equivalent to what has been obtained from traditional Relativity theory [2 Chpt. 12, 3 pp. 520 - 523, 4 pp. 248 - 250, 5 Sect. 15].

Differential Point-To-Point Proper Time

In traditional Relativity theory, Lorentz "proper time" is a "time-like" parameter that is invariant in reference frames translating relative to one-another [3 pp. 519, 5 pp. Sect. 12]. The equivalent for Point-to-Point Relativity derives directly from (46). To expedite the derivation process, it is convenient to reintroduce the $\underline{u}_v$ terminology in (19) for the velocity vector $\underline{v}_{ab/a} = v_{ab}\,\underline{u}_v$. Then with $\alpha = \sqrt{1-v_{ab}^2 / c^2}$ from (34) and $1-\alpha^2 = v_{ab}^2 / c^2$ from (33), the first two rows of (19) become

$$\begin{aligned} d\underline{x}_{p/a} &= d\underline{x}_{p/b} + v_{ab}\, dt_b\, \underline{u}_v + \left(\frac{1}{\sqrt{1-v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/b} \cdot \underline{u}_v + v_{ab}\, dt_b \right) \underline{u}_v \\ dt_a &= \frac{1}{\sqrt{1-v_{ab}^2 / c^2}} \left(dt_b + d\underline{x}_{p/b} \cdot \underline{u}_v\, v_{ab} / c^2 \right) \end{aligned} \qquad (58)$$

As with traditional Relativity [3 pp. 519, 5 pp. Sect. 12], Differential Point-to-Point Relativity proper time is based on its squared value:

$$d\tau^2 \equiv dt^2 - d\underline{x}_p \cdot d\underline{x}_p / c^2 \qquad (59)$$

where

$d\tau$ = Point-to-Point differential proper time interval.

dt = Differential time interval measured on a traditional local clock without particular observer specification (dt_a or dt_b).

$d\underline{x}_p$ = Differential changes in point *p* position vector over the dt differential time interval without particular observer specification ($d\underline{x}_{p/a}$ or $d\underline{x}_{p/b}$).

Note that (59) is similar to the equivalent for traditional Relativity in which proper time is defined as a differential time change function of differential changes in measured distance and time. Similar to traditional Relativity, it will now be shown that Differential Point-to-Point proper time as defined in (59) is the same (i.e., invariant) between observers *a* and *b* translating relative to one another.

For observer a, proper time $d\tau$ calculates from (59) as

$$d\tau^2 = dt_a^2 - d\underline{x}_{p/a} \cdot d\underline{x}_{p/a} / c^2 \tag{60}$$

The differential terms in (60) derive from (58). For the derivation, it is first useful to expand $d\underline{x}_{p/a}$ and $d\underline{x}_{p/b}$ into components parallel and perpendicular to $\underline{u}_v$:

$$d\underline{x}_{p/a} = d\underline{x}_{p/a_v} + d\underline{x}_{p/a_\perp} \qquad d\underline{x}_{p/b} = d\underline{x}_{p/b_v} + d\underline{x}_{p/b_\perp} \tag{61}$$

where

$d\underline{x}_{p/a_v}$, $d\underline{x}_{p/b_v}$ = Components of $d\underline{x}_{p/a}$, $d\underline{x}_{p/b}$ parallel to $\underline{u}_v$.

$d\underline{x}_{pq/a_\perp}$, $d\underline{x}_{p/b_\perp}$ = Components of $d\underline{x}_{p/a}$, $d\underline{x}_{p/b}$ perpendicular to $\underline{u}_v$.

With (61), (58) becomes

$$\begin{aligned} d\underline{x}_{p/a} &= d\underline{x}_{p/b_\perp} + \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\left(d\underline{x}_{p/b_v} + v_{ab}\, dt_b\, \underline{u}_v\right) \\ dt_a &= \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\left(dt_b + d\underline{x}_{p/b_v} \cdot \underline{u}_v\, v_{ab} / c^2\right) \end{aligned} \tag{62}$$

Note also that from the definition of the (61) components:

$$\begin{aligned} d\underline{x}_{p/a_v} &= d\underline{x}_{p/a} \cdot \underline{u}_v\, \underline{u}_v \qquad & d\underline{x}_{p/b_v} &= d\underline{x}_{p/b} \cdot \underline{u}_v\, \underline{u}_v \\ d\underline{x}_{p/a_v} \cdot d\underline{x}_{p/a_v} &= \left(d\underline{x}_{p/a} \cdot \underline{u}_v\right)^2 \qquad & d\underline{x}_{p/b_v} \cdot d\underline{x}_{p/b_v} &= \left(d\underline{x}_{p/b} \cdot \underline{u}_v\right)^2 \end{aligned} \tag{63}$$

$$\begin{aligned} d\underline{x}_{p/a} \cdot d\underline{x}_{p/a} &= d\underline{x}_{p/a_v} \cdot d\underline{x}_{p/a_v} + d\underline{x}_{p/a_\perp} \cdot d\underline{x}_{p/a_\perp} \\ d\underline{x}_{p/b} \cdot d\underline{x}_{p/b} &= d\underline{x}_{p/b_v} \cdot d\underline{x}_{p/b_v} + d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp} \end{aligned} \tag{64}$$

The dt_a^2 and $d\underline{x}_{p/a} \cdot d\underline{x}_{p/a}$ terms in (60) are from (62) with (63) for $\left(d\underline{x}_{p/b_v} \cdot \underline{u}_v\right)^2$:

$$d\underline{x}_{p/a} \cdot d\underline{x}_{p/a} =$$

$$\left[\begin{matrix} d\underline{x}_{p/b_\perp} \\ +\dfrac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\begin{matrix} d\underline{x}_{p/b_v} \\ +v_{ab}\, dt_b\, \underline{u}_v \end{matrix}\right)\end{matrix}\right] \cdot \left[\begin{matrix} d\underline{x}_{p/b_\perp} \\ +\dfrac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\begin{matrix} d\underline{x}_{pq/b_v} \\ +v_{ab}\, dt_b\, \underline{u}_v \end{matrix}\right)\end{matrix}\right]$$

$$= d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp} + \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left[\begin{matrix} d\underline{x}_{p/b_\upsilon} \cdot d\underline{x}_{p/b_v} \\ +2\, d\underline{x}_{p/b_v} \cdot \underline{u}_v\, v_{ab}\, dt_b + v_{ab}^2\, dt_b^2 \end{matrix}\right] \tag{65}$$

$$dt_a^2 = \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left(dt_b + d\underline{x}_{p/b_v} \cdot \underline{u}_\upsilon\, v_{ab}/c^2\right)^2$$

$$= \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left[dt_b^2 + 2\, d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_v\, v_{ab}\, dt_b / c^2 + \left(d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_v\right)^2 v_{ab}^2 / c^4\right]$$

Substituting (65) in (60) then finds for Differential Point-to-Point Relativity proper time:

$$d\tau^2 = \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left[dt_b^2 + 2\, d\underline{x}_{p/b_v} \cdot \underline{u}_v\, v_{ab}\, dt_{pq/b} / c^2 + d\underline{x}_{p/b_v} \cdot d\underline{x}_{p/b_v}\, v_{ab}^2 / c^4\right]$$

$$- d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp} / c^2$$

$$- \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left[d\underline{x}_{p/b_v} \cdot d\underline{x}_{p/b_v} + 2\, d\underline{x}_{p/b_v} \cdot \underline{u}_v\, v_{ab}\, dt_b + v_{ab}^2\, dt_b^2\right] / c^2 \tag{66}$$

$$= \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left[\left(1-v_{ab}^2/c^2\right) dt_b^2 + \left(v_{ab}^2/c^4 - 1/c^2\right) d\underline{x}_{p/b_v} \cdot d\underline{x}_{p/b_v}\right]$$

$$- d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp} / c^2$$

$$= \Delta t_b^2 - \left(d\underline{x}_{p/b_v} \cdot d\underline{x}_{p/b_v} - d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp}\right) / c^2$$

or with (64) and (60):

$$d\tau^2 = dt_a^2 - d\underline{x}_{p/a} \cdot d\underline{x}_{p/a} / c^2 = dt_b^2 - d\underline{x}_{p/b} \cdot d\underline{x}_{p/b} / c^2 \tag{67}$$

Eq. (67) demonstrates the invariance of Differential Point-to-Point proper time formula (59) as determined by observer *a* or by observer *b*. The (67) results are equivalent to what has been obtained with traditional Relativity theory [5 pp. 519, 5 Sect. 12].

Eq. (67) can also be used to show the relationship between proper time and the time differential measured on the point *a* and *b* clocks. From (67),

$$d\tau^2 = dt_a^2 - d\underline{x}_{p/a} \cdot d\underline{x}_{p/a} / c^2 = \left(1 - \frac{d\underline{x}_{p/a}}{dt_a} \cdot \frac{d\underline{x}_{p/a}}{dt_a} / c^2\right) dt_a^2 \tag{68}$$

Then, from (68), the point p velocity definitions in (38), and the equivalent for the observer b:

$$dt_a = d\tau / \sqrt{1 - \underline{v}_{p/a} \cdot \underline{v}_{p/a} / c^2} \qquad dt_b = d\tau / \sqrt{1 - \underline{v}_{p/b} \cdot \underline{v}_{p/b} / c^2} \tag{69}$$

Eqs. (69) also show that:

$$\frac{dt_a}{dt_b} = \sqrt{\frac{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{p/b} / c^2\right)}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{p/a} / c^2\right)}} \tag{70}$$

CONCLUSIONS

Basing Point-To-Point Relativity kinematics on observations of differential remote point position change enables the relative acceleration between observation points to be analytically accommodated. Although not specifically stated, the Point-To-Point Relativity vector formulas derived in this article are only valid in "inertially non-rotating systems" (as in traditional Relativity theory). The differential form of the resulting equations will allow their direct expansion into Point-To-Point kinematics in rotating coordinate systems in a planned future article.

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