# DIGITAL INTEGRATION ALGORITHM ERROR FOR BAND - LIMITED RANDOM PROCESS INPUTS

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ABSTRACT

This article develops an analytical model for evaluating digital integration algorithm error build-up under band-limited random process input. It is demonstrated that the digital integration process introduces a random walk type error in the output that is directly proportional to the root-mean-square input amplitude, directly proportional to the square-root of the input bandwidth, and inversely proportional to the digital integration update frequency.

## ANALYSIS

Consider the analytical model in Fig. 1, illustrating a band limited random noise process to be integrated. A true integral is illustrated in the top half of Fig. 1. An approximate digital integration process is described in the lower half of Fig. 1 as a trapezoidal integration algorithm with updating time interval T and cycle index n. The digital integration error is the difference between the digital and true integrals. The following analysis develops a simple equation for evaluating the digital integration error as a function of the integration algorithm processing rate and the band-limited random noise magnitude/bandwidth.

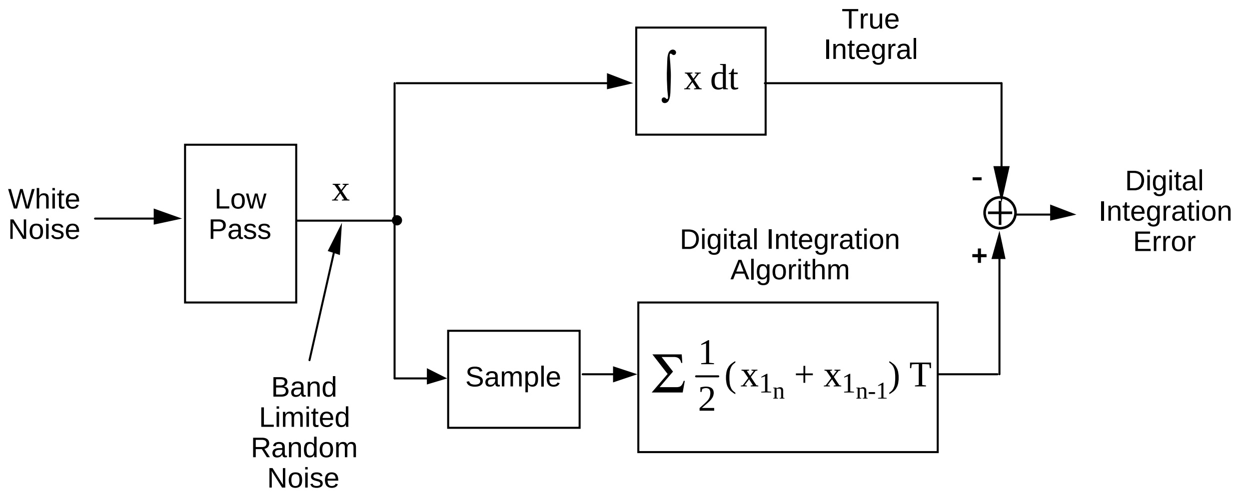


Fig. 1 - Digital Integration Algorithm Error

The analysis begins by first describing the Fig. 1 model analytically:

$$\dot{x}_1 = -\omega_{bw}\, x_1 + n_{p_1} \qquad \dot{x}_2 = x_1 \tag{1}$$

$$x_{3_n} = x_{3_{n-1}} + \frac{1}{2}\left(x_{1_n} + x_{1_{n-1}}\right) T \tag{2}$$

where,

$x_1$ = Band-limited white noise input to the integrators (modeled as a first order Markov process).

$\omega_{bw}$ = Bandwidth of band-limited white noise inputs (i.e., Markov process gain).

$n_{p_1}$ = White noise input to the Markov process.

$x_2$ = True integral of $x_1$.

$x_3$ = Digital integral of $x_1$ (using a trapezoidal integration algorithm).

$n$ = Integration algorithm computation cycle index.

$T$ = Integration algorithm computation cycle time period (reciprocal of algorithm iteration rate).

Eqs. (1) can also be expressed in the equivalent state vector form [1, Eq. (15-2)]:

$$\underline{\dot{x}} = A\, \underline{x} + \underline{n}_p \tag{3}$$

in which

$$\underline{x} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A \equiv \begin{bmatrix} -\omega_{bw} & 0 \\ 1 & 0 \end{bmatrix} \qquad \underline{n}_p \equiv \begin{bmatrix} n_{p_1} \\ 0 \end{bmatrix} \tag{4}$$

and where

$\underline{x}$ = State vector.

$A$ = State dynamic matrix.

$\underline{n}_p$ = Process noise vector.

The equivalent integrated form of (3), valid at the digital integration algorithm iteration times, is given by the classical form [1, Eq. (15.1.1-14)]:

$$\underline{x}_n = \Phi_n\, \underline{x}_{n-1} + \underline{q}_n \tag{5}$$

in which

$$\Phi_n = \Phi(T) \qquad \Phi(\tau) = \int_{t_{n-1}}^{t_{n-1}+\tau} A\,\Phi(\tau_1)\,d\tau_1 \qquad \Phi(0) = I$$

$$\underline{q}_n = \int_{t_{n-1}}^{t_{n-1}+T} \Phi(\tau)\,\underline{n}_P\,d\tau \tag{6}$$

where

$\Phi$ = State transition matrix.

I = Identity matrix.

$\underline{q}_n$ = Integrated process noise vector (over $t_{n-1}$ to $t_n$).

It is easily shown by integrating (6) for $\Phi(\tau)$ that

$$\Phi_n = \begin{bmatrix} e^{-\omega_{bw}T} & 0 \\ \dfrac{1}{\omega_{bw}}\left(1 - e^{-\omega_{bw}T}\right) & 1 \end{bmatrix} \tag{7}$$

We now define

$$\Phi_n \equiv \begin{bmatrix} \phi 11_n & \phi 12_n \\ \phi 21_n & \phi 22_n \end{bmatrix} \qquad \underline{q}_n \equiv \begin{bmatrix} q1_n \\ q2_n \end{bmatrix} \tag{8}$$

From (7):

$$\phi 11_n = e^{-\omega_{bw}T} \qquad \phi 12_n = 0$$

$$\phi 21_n = \frac{1}{\omega_{bw}}\left(1 - \phi 11_n\right) \qquad \phi 22_n = 1 \tag{9}$$

With (8) and (9), (5) and (2) become in component form:

$$x_{1n} = \phi 11_n\, x1_{n-1} + q1_n$$

$$x2_n = x2_{n-1} + \phi 21_n\, x1_{n-1} + q2_n \tag{10}$$

$$x3_n = x3_{n-1} + \frac{1}{2}\left(x1_n + x1_{n-1}\right) T$$

The error in $x_3$ is defined as the difference between $x_3$ and the true integral $x_2$:

$$e_n \equiv x_{3_n} - x_{2_n} \tag{11}$$

where

$e_n$ = Error in $x_{3_n}$.

The $e_n$ error can be evaluated in terms of its variance:

$$P_{ee_n} = E\left(e_n^2\right) = E\left[\left(x_{3_n} - x_{2_n}\right)^2\right] = E\left(x_{3_n}^2\right) + E\left(x_{2_n}\right)^2 - 2\,E\left(x_{3_n}\, x_{2_n}\right) \tag{12}$$

or

$$P_{ee_n} = P_{33_n} + P_{22_n} - 2\,P_{32_n} \tag{13}$$

where

$E(\;)$ = Expected value operator.

$P_{ee_n}$ = Variance of $e_n$.

$P_{ij_n}$ = Expected value of $x_{i_n}\, x_{j_n}$ (i.e., the covariance of $x_{i_n}$ with $x_{j_n}$).

For error build-up analysis, it is convenient to define (13) in terms of its build-up rate over a computer cycle:

$$\dot{P}_{ee_n} \equiv \Delta P_{ij_n} / T = \left(\Delta P_{33_n} + \Delta P_{22_n} - 2\,\Delta P_{32_n}\right) / T \tag{14}$$

in which

$$\Delta P_{ij_n} \equiv P_{ij_n} - P_{ij_{n-1}} \tag{15}$$

and where

$\dot{P}_{ee_n}$ = Error e variance build-up rate over computer cycle n.

We now develop analytical expressions for the individual terms in (14) by first forming particular products of the (10) terms:

$$x_{3_n} x_{1_n} = \left[x_{3_{n-1}} + \frac{1}{2}\left(x_{1_n} + x_{1_{n-1}}\right) T\right] x_{1_n} = x_{3_{n-1}} x_{1_n} + \frac{1}{2} x_{1_n}^2 T + \frac{1}{2} x_{1_{n-1}} x_{1_n} T$$

$$x_{3_n} x_{2_n} = \left[x_{3_{n-1}} + \frac{1}{2}\left(x_{1_n} + x_{1_{n-1}}\right) T\right] x_{2_n} = x_{3_{n-1}} x_{2_n} + \frac{1}{2} x_{1_n} x_{2_n} T + \frac{1}{2} x_{1_{n-1}} x_{2_n} T$$

$$x_{3_n}^2 = \left[x_{3_{n-1}} + \frac{1}{2}\left(x_{1_n} + x_{1_{n-1}}\right) T\right]^2 \tag{16}$$

$$= x_{3n-1}^2 + x_{3_{n-1}}\left(x_{1_n} + x_{1_{n-1}}\right) T + \frac{1}{4}\left(x_{1_n}^2 + 2\, x_{1_n} x_{1_{n-1}} + x_{1_{n-1}}^2\right) T^2$$

$$= x_{3n-1}^2 + x_{3_{n-1}} x_{1_{n-1}} T + \frac{1}{4} x_{1_n}^2 T^2 + \frac{1}{4} x_{1_{n-1}}^2 T^2 + x_{3_{n-1}} x_{1_n} T + \frac{1}{2} x_{1_n} x_{1_{n-1}} T^2$$

Particular terms in (16) are with (10):

$$x_{3_{n-1}} x_{1_n} = x_{3_{n-1}}\left(\phi_{11_n} x_{1_{n-1}} + q_{1_n}\right) = \phi_{11_n} x_{3_{n-1}} x_{1_{n-1}} + x_{3_{n-1}} q_{1_n}$$

$$x_{1_{n-1}} x_{1_n} = x_{1_{n-1}}\left(\phi_{11_n} x_{1_{n-1}} + q_{1_n}\right) = \phi_{11_n} x_{1_{n-1}}^2 + x_{1_{n-1}} q_{1_n}$$

$$x_{3_{n-1}} x_{2_n} = x_{3_{n-1}}\left(x_{2_{n-1}} + \phi_{21_n} x_{1_{n-1}} + q_{2_n}\right) = x_{3_{n-1}} x_{2_{n-1}} + \phi_{21_n} x_{3_{n-1}} x_{1_{n-1}} + x_{3_{n-1}} q_{2_n} \tag{17}$$

$$x_{1_{n-1}} x_{2_n} = x_{1_{n-1}}\left(x_{2_{n-1}} + \phi_{21_n} x_{1_{n-1}} + q_{2_n}\right) = x_{1_{n-1}} x_{2_{n-1}} + \phi_{21_n} x_{1_{n-1}}^2 + x_{1_{n-1}} q_{2_n}$$

Substituting (17) in (16), taking the expected value, and recognizing that the $q_n$ terms are uncorrelated with $x_{n-1}$ terms yields:

$$P_{31_n} = \phi_{11_n} P_{31_{n-1}} + \frac{1}{2} P_{11_n} T + \frac{1}{2} \phi_{11_n} P_{11_{n-1}} T$$

$$P_{32_n} = P_{32_{n-1}} + \phi_{21_n} P_{31_{n-1}} + \frac{1}{2} P_{12_n} T + \frac{1}{2} P_{12_{n-1}} T + \phi_{21_n} P_{11_{n-1}} T \tag{18}$$

$$P_{33_n} = P_{33_{n-1}} + \left(1 + \phi_{11_n}\right) P_{31_{n-1}} T + \frac{1}{4} P_{11_n} T^2 + \left(\frac{1}{4} + \frac{1}{2} \phi_{11_n}\right) P_{11_{n-1}} T^2$$

The $P_{11}$ and $P_{12}$ terms in (18) are evaluated from the continuous covariance form of (3) [1, Eq. (15.1.2.1.1.3-1)] :

$$\dot{P} = A\, P + P\, A^T + N_p \tag{19}$$

in which

$$P \equiv E\left(\underline{x}\ \underline{x}^T\right) \equiv \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad N_p \equiv \begin{bmatrix} N_{p_{11}} & 0 \\ 0 & 0 \end{bmatrix} \tag{20}$$

and where

$P_{ij}$ = Element in row i column j of covariance matrix P.

$N_p$ = Process noise density matrix.

$N_{p_{11}}$ = Density of process noise $n_{p_1}$.

Expanding (19) with (20) and (3) in component form yields:

$$\dot{P}_{11} = -2\,\omega_{bw}\,P_{11} + N_{p_{11}} \qquad \dot{P}_{12} = \dot{P}_{21} = P_{11} - \omega_{BW}\,P_{12} \qquad \dot{P}_{22} = 2\,P_{12} \tag{21}$$

We now assume that at the start of the Fig. 1 integration process, $P_{11}$ is at its steady state value. Then from (21):

$$\dot{P}_{12} + \omega_{bw}\,P_{12} = P_{11_\infty} \qquad \dot{P}_{22} = 2\,P_{12} \tag{22}$$

where,

$P_{11_\infty}$ = Steady state value of $P_{11}$.

The integral solution to (22) starting with zero $P_{12}$ and $P_{22}$ initial conditions is

$$P_{12} = \frac{1}{\omega_{bw}}\left(1 - e^{-\omega_{bw}\,t}\right) P_{11_\infty} \qquad P_{22} = \frac{2}{\omega_{bw}}\left[t - \frac{1}{\omega_{bw}}\left(1 - e^{-\omega_{bw}\,t}\right)\right] P_{11_\infty} \tag{23}$$

where

t = Time since start of the Fig. 1 integration process.

At computer cycles n-1 and n, $P_{22}$ in (23) is given by:

$$\begin{aligned} P_{22_{n-1}} &= \frac{2}{\omega_{bw}}\left[t_{n-1} - \frac{1}{\omega_{bw}}\left(1 - e^{-\omega_{bw}\,t_{n-1}}\right)\right] P_{11_\infty} \\ P_{22_n} &= \frac{2}{\omega_{bw}}\left[t_n - \frac{1}{\omega_{bw}}\left(1 - e^{-\omega_{bw}\,T}\,e^{-\omega_{bw}\,t_{n-1}}\right)\right] P_{11_\infty} \end{aligned} \tag{24}$$

or

$$P_{22_n} = P_{22_{n-1}} + \frac{2}{\omega_{bw}}\left[T - \left(1 - e^{-\omega_{bw}\,T}\right) e^{-\omega_{bw}\,t_{n-1}}\right] P_{11_\infty} \tag{25}$$

in which from its definition

$$T = t_n - t_{n-1} \tag{26}$$

From (23) at cycle n, $P_{12}$ is

$$P_{12_n} = \frac{1}{\omega_{bw}}\left(1 - e^{-\omega_{bw}\ t_n}\right) P_{11_\infty} \tag{27}$$

We also note that

$$P_{11_n} = P_{11_{n-1}} = P_{11_\infty} \tag{28}$$

and from (15)

$$\Delta P_{33_n} = P_{33_n} - P_{33_{n-1}} \qquad \Delta P_{22_n} = P_{22_n} - P_{22_{n-1}} \qquad \Delta P_{32_n} = P_{32_n} - P_{32_{n-1}} \tag{29}$$

Eqs. (18) with (25), (27), (28) and (29) comprise a complete set for evaluating $\dot{P}_{ee_n}$ in (14).

The analysis is now simplified by considering the steady state solution to (14) (i.e., for very large n). In the steady state we write from (18) and (27):

$$\begin{gathered} P_{31_n} = P_{31_{n-1}} = P_{31_\infty} \qquad P_{12_n} = P_{12_{n-1}} = P_{12_\infty} \\ P_{31_\infty} = \phi_{11_n} P_{31_\infty} + \frac{1}{2}\left(1 + \phi_{11_n}\right) P_{11_\infty}\, T \qquad P_{12_\infty} = \frac{1}{\omega_{bw}}\left(1 - e^{-\omega_{bw}\, t_\infty}\right) P_{11_\infty} \end{gathered} \tag{30}$$

or

$$\begin{gathered} P_{12_n} = \frac{1}{\omega_{bw}} P_{11_\infty} \\ P_{31_n} = P_{31_{n-1}} = P_{31_\infty} = \frac{1}{2}\left(\frac{1 + \phi_{11_n}}{1 - \phi_{11_n}}\right) P_{11_\infty}\, T \end{gathered} \tag{31}$$

where,

$P_{12_\infty}$, $P_{31_\infty}$ = Steady state values for $P_{12}$ and $P_{31}$.

Then with (18), (25), (28) and (31), (29) becomes

$$\Delta P_{32_n} = \phi_{21_n} P_{31_\infty} + P_{12_\infty} T + \frac{1}{2} \phi_{21_n} P_{11_\infty} T$$

$$= \left[\frac{1}{2} \phi_{21_n} \left(\frac{1 + \phi_{11_n}}{1 - \phi_{11_n}}\right) + \frac{1}{2} \phi_{21_n} + \frac{1}{\omega_{bw}}\right] P_{11_\infty} T = \left[\frac{\phi_{21_n}}{1 - \phi_{11_n}} + \frac{1}{\omega_{bw}}\right] P_{11_\infty} T$$

(32)

$$\Delta P_{33_n} = \left(1 + \phi_{11_n}\right) P_{31_\infty} T + \frac{1}{2} \left(1 + \phi_{11_n}\right) P_{11_\infty} T^2$$

$$= \frac{1}{2} \left(1 + \phi_{11_N}\right) \left[\frac{1 + \phi_{11_n}}{1 - \phi_{11_n}} + 1\right] P_{11_\infty} T^2 = \left(\frac{1 + \phi_{11_n}}{1 - \phi_{11_n}}\right) P_{11_\infty} T^2$$

$$\Delta P_{22_n} = \frac{2}{\omega_{bw}} P_{11_\infty} T$$

Substituting (32) into (14) yields

$$\dot{P}_{ee_n} = \left[\left(\frac{1 + \phi_{11_n}}{1 - \phi_{11_n}}\right) T + \frac{2}{\omega_{bw}} - 2 \left(\frac{\phi_{21_n}}{1 - \phi_{11_n}} + \frac{1}{\omega_{bw}}\right)\right] P_{11_\infty} = \left[\frac{\left(1 + \phi_{11_n}\right) T - 2 \phi_{21_n}}{1 - \phi_{11_n}}\right] P_{11_\infty} \qquad (33)$$

From (9):

$$\frac{\left(1 + \phi_{11_n}\right) T - 2 \phi_{21_n}}{1 - \phi_{11_n}} = \frac{1 + \phi_{11_n} - \frac{2}{\omega_{bw}} \left(1 - \phi_{11_n}\right)}{1 - \phi_{11_n}}$$

(34)

$$= \left(\frac{1 + \phi_{11_n}}{1 - \phi_{11_n}}\right) T - \frac{2}{\omega_{bw}} = \left(\frac{1 + e^{-\omega_{bw} T}}{1 - e^{-\omega_{bw} T}} - \frac{2}{\omega_{bw} T}\right) T$$

Finally, from (33) and (34):

$$\sigma_{e_{rw}} = \sigma_x \sqrt{\left(\frac{1 + e^{-\mu}}{1 - e^{-\mu}} - \frac{2}{\mu}\right) \frac{1}{f_c}} \qquad (35)$$

where

$\sigma_{e_{rw}} = \sqrt{\dot{P}_{ee_n}}$ = Standard deviation of integration algorithm random walk output error (per square-root-of-secs).

$f_c = 1 / T$ = Integration algorithm iteration rate (hz).

$\sigma_x = \sqrt{P_{11_\infty}}$ = Standard deviation of band-limited input to digital integrator.

$\mu = \omega_{bw} / f_c$ = Bandwidth/iteration rate ratio (rad/sec per hz).

Eq. (35) is an exact expression for the steady state random walk error in the digital integration process. An approximate expression can be obtained for $\sigma_{e_{rw}}$ for small values of $\mu$ from the Taylor series expansion:

$$e^{-\mu} = 1 - \mu + \frac{1}{2}\mu^2 - \frac{1}{6}\mu^3 + ... \tag{36}$$

From (36):

$$\begin{aligned}
\frac{1 + e^{-\mu}}{1 - e^{-\mu}} - \frac{2}{\mu} &= \frac{2 - \mu + \frac{1}{2}\mu^2 - \frac{1}{6}\mu^3 + ...}{\mu - \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3 - ...} - \frac{2}{\mu} = \frac{2}{\mu}\frac{\left(1 - \frac{1}{2}\mu + \frac{1}{4}\mu^2 - \frac{1}{6}\mu^3 + ...\right)}{\left(1 - \frac{1}{2}\mu + \frac{1}{6}\mu^2 - ...\right)} - \frac{2}{\mu} \\
&= \frac{2}{\mu}\left[\frac{1 - \frac{1}{2}\mu + \frac{1}{4}\mu^2 - ... - 1 + \frac{1}{2}\mu - \frac{1}{6}\mu^2 + ...}{1 - \frac{1}{2}\mu + \frac{1}{6}\mu^2 - ...}\right] \approx \frac{2}{\mu}\left(\frac{1}{4} - \frac{1}{6}\right)\mu^2 = \frac{1}{6}\mu
\end{aligned} \tag{37}$$

With (37), (35) becomes

$$\sigma_{e_{rw}} \approx \sigma_x \sqrt{\frac{1}{6}\frac{\mu}{f_c}} \tag{38}$$

Substituting for the definition of $\mu \equiv \omega_{bw} / f_c$ then yields the approximate form:

$$\sigma_{e_{rw}} \approx \sqrt{\frac{1}{6}\omega_{bw}}\;\frac{1}{f_c}\,\sigma_x \tag{39}$$

REFERENCE

[1] Savage, P. G., *Strapdown Analytics, Second Edition*, Strapdown Associates, Inc., 2007