FIXED GAIN DIGITAL FILTER DESIGN FOR SPECIFIED PHASE VERSUS FREQUENCY RESPONSE

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ABSTRACT

A new concept in non-recursive fixed gain digital filter design is presented to achieve a specified dynamic phase shift response over a user defined frequency range. The desired response can be specified analytically or numerically. Filter coefficients are designed based on an optimal formulation to minimize the integral of a filter error parameter. The minimization process can be performed numerically for a general error parameter selection, or analytically based on a minimum least squares formulation. Numerical examples are provided showing how the filter can be utilized to eliminate phase shift from a digitally sampled analog sensor.

INTRODUCTION

Fixed gain digital filter design has generally been based on creating a filter amplitude response to sinusoidal inputs that realizes some general performance criteria (e.g., generating an equivalent pole/zero configuration as a reference analog filter, approximating a specified analog transfer function using a bilinear transformation process, generating a specified amplitude response at selected discrete frequencies, generating a maximally flat amplitude response [1 - Sects. 6.3.1, 6.4.2, 7.3, 7.6]. Digital filter design for a specified phase response has generally only been utilized as a means of simplifying the form of an amplitude designed digital filter algorithm (e.g., linear phase response though symmetrical coefficients - [1 - Sect 7.2]) or as the design approach for a general coefficient configuration (e.g., minimum phase - [1 - Sect. 7.5.5]). This paper introduces a new concept in constant coefficient digital filter design to achieve a specified phase response over a user specified frequency range. The specified response can be defined as an analytical or numerical function of input frequency. The filter configuration is a simple non-recursive weighted sum of current and past filter inputs over a selected moving time window.

This article is a generalized and expanded version of the same approach first published by the author in Appendix B of Reference [2].

FILTER DESIGN FOR SPECIFIED DYNAMIC PHASE RESPONSE

The digital filter structure selected for this application is of the general finite impulse response (FIR) non-recursive form:

$$y(t_n) = \sum_{i=1}^{M} b_i \, x\left[t_n - (i-1)\,T\right] \tag{1}$$

where y is the filter output, x is the filter input, T is the time between filter input (and output) samples, t_n is the time at filter (and input) computer cycle n, and M is the number of current through past input samples used for each $y(t_n)$ output cycle. The non-recursive (1) form was selected to intentionally exclude past values of y as part of the $y(t_n)$ input to guarantee a known finite response time and to eliminate potential undesirable intrinsic dynamic characteristics from the resulting filter response (e.g., marginally stable roots).

Under a unit amplitude digital sinusoidal input $x(t_n)$ in (1), each i^{th} contribution to $y(t_n)$ is:

$$\begin{aligned} x\left[t_n - (i-1)\,T\right] &= \sin\left\{\Omega\left[t_n - (i-1)\,T\right]\right\} \\ &= \sin\left(\Omega\, t_n\right)\cos\left[\Omega(i-1)\,T\right] - \cos\left(\Omega\, t_n\right)\sin\left[\Omega(i-1)\,T\right] \end{aligned} \tag{2}$$

so that (1) becomes:

$$y(t_n) = \sin\left(\Omega\, t_n\right) \sum_{i=1}^{M} b_i \cos\left[\Omega(i-1)\,T\right] - \cos\left(\Omega\, t_n\right) \sum_{i=1}^{M} b_i \sin\left[\Omega(i-1)\,T\right] \tag{3}$$

The (3) response can also be represented by the general form:

$$\begin{aligned} y(t_n) &= A_{Flt}(\Omega)\, \sin\left[\left(\Omega\, t_n\right) + \gamma_{Flt}(\Omega)\right] \\ &= \sin\left(\Omega\, t_n\right)\, A_{Flt}(\Omega)\, \cos\left[\gamma_{Flt}(\Omega)\right] + \cos\left(\Omega\, t_n\right)\, A_{Flt}(\Omega)\, \sin\left[\gamma_{Flt}(\Omega)\right] \end{aligned} \tag{4}$$

Comparing (3) and (4) shows the equivalency

$$\begin{aligned} A_{Flt}(\Omega)\, \cos\left[\gamma_{Flt}(\Omega)\right] &= \sum_{i=1}^{M} b_i \cos\left[\Omega(i-1)\,T\right] \\ A_{Flt}(\Omega)\, \sin\left[\gamma_{Flt}(\Omega)\right] &= -\sum_{i=1}^{M} b_i \sin\left[\Omega(i-1)\,T\right] \end{aligned} \tag{5}$$

The filter amplitude $A_{Flt}(\Omega)$ and phase response $\gamma_{Flt}(\Omega)$ can be derived from (5) as:

$$\begin{aligned} &\left\{A_{Flt}(\Omega)\, \cos\left[\gamma_{Flt}(\Omega)\right]\right\}^2 + \left\{A_{Flt}(\Omega)\, \sin\left[\gamma_{Flt}(\Omega)\right]\right\}^2 = A_{Flt}(\Omega)^2 \\ &\qquad = \left\{\sum_{i=1}^{M} b_i \cos\left[\Omega(i-1)\,T\right]\right\}^2 + \left\{\sum_{i=1}^{M} b_i \sin\left[\Omega(i-1)\,T\right]\right\}^2 \end{aligned}$$

$$\tan\left[\gamma_{Flt}(\Omega)\right] = \frac{A_{Flt}(\Omega)\, \sin\left[\gamma_{Flt}(\Omega)\right]}{A_{Flt}(\Omega)\, \cos\left[\gamma_{Flt}(\Omega)\right]} = -\frac{\sum_{i=1}^{M} b_i \sin\left[\Omega(i-1)\,T\right]}{\sum_{i=1}^{M} b_i \cos\left[\Omega(i-1)\,T\right]}$$

hence,

$$A_{Flt}(\Omega) = \sqrt{\left\{\sum_{i=1}^{M} b_i \cos\left[\Omega(i-1)\,T\right]\right\}^2 + \left\{\sum_{i=1}^{M} b_i \sin\left[\Omega(i-1)\,T\right]\right\}^2}$$

$$\gamma_{Flt}(\Omega) = -\tan^{-1}\left\{\frac{\sum_{i=1}^{M} b_i \sin\left[\Omega(i-1)\,T\right]}{\sum_{i=1}^{M} b_i \cos\left[\Omega(i-1)\,T\right]}\right\} \qquad (6)$$

The filter design requirement being addressed is for $\gamma_{Flt}(\Omega)$ in (6) to match a specified $\gamma_{Desired}(\Omega)$ over a specified frequency range Ω_{Range}. This can be achieved in general, by first defining a general error parameter $h(\Omega)$ as a function of the magnitude of the difference between $\gamma_{Flt}(\Omega)$ and $\gamma_{Desired}(\Omega)$:

$$h(\Omega) \equiv f\left\{\left|\,\gamma_{Flt}(\Omega) - \gamma_{Desired}(\Omega)\right|\right\} \qquad f(0) = \mathbf{0} \qquad (7)$$

The (6) b_i coefficients would then be calculated as those that minimize the average value of $h(\Omega)$ over Ω_{Range}, represented analytically as the minimization of z, the integral of $h(\Omega)$ over Ω_{Range}:

$$z \equiv \int_{\Omega_{Range}} h(\Omega)\, d\Omega \qquad (8)$$

When feasible, the z minimization process can be performed analytically. Alternatively, z minimization can always be achieved numerically by evaluating z over all possible b_i values and saving the bi s for minimum z along the way. For current high-speed computer technology, the latter would yield the desired solution in relatively short order, depending on the number of terms M carried in (1).

When z minimization is analytically achievable, classical methods can be utilized (e.g., a minimum least-squares integral approach) as in [3] for strapdown inertial navigation system coning algorithm coefficients, or in [1 - Sect. 7.4] for non-recursive fixed gain filters designed to meet a specified amplitude versus frequency characteristic. In the latter case, the design was performed using a linear phase versus frequency response filter achieved through use of symmetrical filter coefficients. For the specified phase matching approach presented here, no restriction is required on the general form of the (1) filter b_i coefficients.

In formulating the minimum least-squares phase error design process, an error parameter $e(\Omega)$ is first defined as:

$$
\begin{aligned}
e(\Omega) &\equiv W(\Omega)\begin{Bmatrix} A_{Flt}(\Omega)\cos\left[\gamma_{Flt}(\Omega)\right]\sin\left[\gamma_{Desired}(\Omega)\right] \\ -A_{Flt}(\Omega)\sin\left[\gamma_{Flt}(\Omega)\right]\cos\left[\gamma_{Desired}(\Omega)\right] \end{Bmatrix} \\
&= -W(\Omega)\,A_{Flt}(\Omega)\sin\left[\gamma_{Flt}(\Omega)-\gamma_{Desired}(\Omega)\right]
\end{aligned}
\tag{9}
$$

where $W(\Omega)$ is a general application dependent weighting factor. Thus, $e(\Omega)$ measures the weighted error in the filter's ability to achieve the goal of setting $\gamma_{Flt}(\Omega)$ equal to $\gamma_{Desired}(\Omega)$. Substituting (5) into (9) finds $e(\Omega)$ in (9) as a function of $\gamma_{Desired}(\Omega)$ and the b_i s:

$$
e(\Omega) = f(\Omega)\sum_{i=1}^{M} b_i\cos\left[\Omega(i-1)T\right] + g(\Omega)\sum_{i=1}^{M} b_i\sin\left[\Omega(i-1)T\right] \tag{10}
$$

where

$$
f(\Omega) \equiv W(\Omega)\sin\left[\gamma_{Desired}(\Omega)\right] \qquad g(\Omega) \equiv W(\Omega)\cos\left[\gamma_{Desired}(\Omega)\right] \tag{11}
$$

An exactness constraint is also incorporated requiring $y(t_n) = x(t_n)$ at low frequencies when $x\left[t_n-(i-1)T\right] \approx x(t_n)$. From (1), this is equivalent to $\sum_{i=1}^{M} b_i = 1$ or:

$$
b_1 = 1 - \sum_{i=2}^{M} b_i \tag{12}
$$

so that (10) becomes

$$
\begin{aligned}
e(\Omega) &= f(\Omega)\left\{1+\sum_{i=2}^{M} b_i\left\{\cos\left[\Omega(i-1)T\right]-1\right\}\right\} + g(\Omega)\sum_{i=2}^{M} b_i\sin\left[\Omega(i-1)T\right] \\
&= f(\Omega) + \sum_{i=2}^{M} b_i\left\langle f(\Omega)\left\{\cos\left[\Omega(i-1)T\right]-1\right\} + g(\Omega)\sin\left[\Omega(i-1)T\right]\right\rangle
\end{aligned}
\tag{13}
$$

Using an integral least-squares minimization approach sets $h(\Omega) = e(\Omega)^2$ in (8) so that with (13):

$$
z = \int_{\Omega_{Range}} \left\{f(\Omega) + \sum_{i=2}^{M} b_i\left\langle f(\Omega)\left\{\cos\left[\Omega(i-1)T\right]-1\right\} + g(\Omega)\sin\left[\Omega(i-1)T\right]\right\rangle\right\}^2 d\Omega \tag{14}
$$

Minimization of (14) can be classically achieved analytically by taking the partial derivative of (14) with respect to each b_i and equating each to zero:

$$
\begin{aligned}
\frac{\partial z}{\partial b_i} &= \frac{\partial \displaystyle\int_{\Omega_{Range}} \left\{f(\Omega) + \sum_{i=2}^{M} b_i\left\langle f(\Omega)\left\{\cos\left[\Omega(i-1)T\right]-1\right\} + g(\Omega)\sin\left[\Omega(i-1)T\right]\right\rangle\right\}^2 d\Omega}{\partial b_i} \\
&= \int_{\Omega_{Range}} e(\Omega)\left\langle f(\Omega)\left\{\cos\left[\Omega(i-1)T\right]-1\right\} + g(\Omega)\sin\left[\Omega(i-1)T\right]\right\rangle d\Omega = 0
\end{aligned}
\tag{15}
$$

Thus, with (13):

$$\int\limits_{\Omega_{Range}} \left\langle \begin{matrix} f(\Omega)\left\{\cos\left[\Omega(i-1)T\right]-1\right\} \\ +g(\Omega)\sin\left[\Omega(i-1)T\right] \end{matrix} \right\rangle \sum_{j=2}^{M} b_j \left\langle \begin{matrix} f(\Omega)\left\{\cos\left[\Omega(j-1)T\right]-1\right\} \\ +g(\Omega)\sin\left[\Omega(j-1)T\right] \end{matrix} \right\rangle d\Omega$$
$$= -\int\limits_{\Omega_{Range}} f(\Omega)\left\langle f(\Omega)\left\{\cos\left[\Omega(i-1)T\right]-1\right\} + g(\Omega)\sin\left[\Omega(i-1)T\right]\right\rangle d\Omega \tag{16}$$

or

$$\sum_{j=2}^{M} b_j \int\limits_{\Omega_{Range}} \left\langle \begin{matrix} f(\Omega)\left\{1-\cos\left[\Omega(i-1)T\right]\right\} \\ -g(\Omega)\sin\left[\Omega(i-1)T\right] \end{matrix} \right\rangle \left\langle \begin{matrix} f(\Omega)\left\{1-\cos\left[\Omega(j-1)T\right]\right\} \\ -g(\Omega)\sin\left[\Omega(j-1)T\right] \end{matrix} \right\rangle d\Omega$$
$$= \int\limits_{\Omega_{Range}} f(\Omega)\left\langle f(\Omega)\left\{1-\cos\left[\Omega(i-1)T\right]\right\} - g(\Omega)\sin\left[\Omega(i-1)T\right]\right\rangle d\Omega \tag{17}$$

In matrix form, (17) is equivalently:

$$A\,U = V \tag{18}$$

where U is an M-1 length column matrix with the element in location r equal to b_{r+1}, A is an M-1 by M-1 square matrix with element $a(r,s)$ in row r, column s equal to

$$a(r,s) = \int\limits_{\Omega_{Range}} \left\{ \begin{matrix} f(\Omega)\left[1-\cos\left(r\,\Omega T\right)\right] \\ -g(\Omega)\sin\left(r\,\Omega T\right) \end{matrix} \right\} \left\{ \begin{matrix} f(\Omega)\left[1-\cos\left(s\,\Omega T\right)\right] \\ -g(\Omega)\sin\left(s\,\Omega T\right) \end{matrix} \right\} d\Omega \tag{19}$$

V is an M-1 length column matrix with element $v(r)$ in location r equal to

$$v(r) = \int\limits_{\Omega_{Range}} f(\Omega)\left\{ f(\Omega)\left[1-\cos\left(r\,\Omega T\right)\right] - g(\Omega)\sin\left(r\,\Omega T\right)\right\} d\Omega \tag{20}$$

and $f(\Omega)$ is as defined in (11). The solution for the best fit b_i s is from (18) with (12):

$$B = \begin{bmatrix} b_1 \\ U \end{bmatrix} \quad \text{with } U = A^{-1}V \quad \text{and} \quad b_1 = 1 - \sum_{r=1}^{M-1} u_r \tag{22}$$

where B is an M length column matrix with element s equal to b_i.

EXAMPLE 1

Using [2] as an example, consider an analog sensor with the following $A_{Sens}(\Omega)$, $\gamma_{Sens}(\Omega)$ dynamic amplitude/phase characteristic as a function of frequency Ω:

$$A_{Sens}(\Omega) = \frac{\omega_{Sens}^2}{\sqrt{\left(\omega_{Sens}^2 - \Omega^2\right)^2 + \left(2\,\varsigma_{Sens}\,\omega_{Sens}\,\Omega\right)^2}} \qquad \gamma_{Sens}(\Omega) = -\tan^{-1}\left[\frac{2\,\varsigma_{Sens}\,\omega_{Sens}\,\Omega}{\left(\omega_{Sens}^2 - \Omega^2\right)}\right] \tag{23}$$

where ω_{Sens} and ς_{Sens} are the undamped natural frequency and damping ratio. The sensor output is sampled into a computer at a 1 KHz sampling rate. For this example, consider a 400 Hz sensor bandwidth ($\omega_{Sens} = 2\pi \times 400$ hz = 2,513 rad/sec) and 0.707 damping ratio ς_{Sens}. It is required (as in [2]) that the $\gamma_{Sens}(\Omega)$ phase shift on the sampled signal be cancelled over a specified frequency range of 0 to 100 Hz using a digital pre-filter.

To cancel $\gamma_{Sens}(\Omega)$, $\gamma_{Desired}(\Omega)$ in (21) would be set to $-\gamma_{Sens}(\Omega)$. To focus filter effectiveness over the specified frequency range, the weighting function $W(\Omega)$ in (21) is set to unity in the 0 - 150 Hz region (spanning the 0 - 100 Hz design specification with a 50 Hz margin), and zero otherwise. For a filter processing rate set to the 1 KHz sensor sample rate (T = 0.001 sec), the least-squares minimization range Ω_{Range} for filter design in (19) and (20) is set to 500 Hz, the Nyquist frequency corresponding to 1 KHz. For such settings in (19), (20) and (23), the resulting (22) filter b_i coefficients based on a 4 sample (M = 1) filter configuration are:

$$b_1 = 1.90452576 \quad b_2 = -1.32175784 \quad b_3 = 0.492638638 \quad b_4 = -0.0754065576 \tag{24}$$

Fig. 1 is a plot of $\gamma_{Sens}(\Omega)$ from (23) and $\gamma_{Flt}(\Omega)$ from (6) with the (24) coefficients, versus frequency (in Hz). Fig. 2 plots $\left|\gamma_{Sens}(\Omega) + \gamma_{Flt}(\Omega)\right|$, the $\gamma_{Flt}(\Omega)$ error in canceling $\gamma_{Sens}(\Omega)$. Fig. 3 shows the corresponding $A_{Sens}(\Omega)$, $A_{Flt}(\Omega)$ amplitude responses from (23) and (6) with the (24) coefficients, and the resulting combined filtered sensor amplitude response $A_{Flt}(\Omega)\,A_{Sens}(\Omega)$. Fig. 2 demonstrates the filter's effectiveness in canceling $\gamma_{Sens}(\Omega)$ over the 0 - 100 Hz design frequency range Ω_{Range}.

EXAMPLE 2

As a second example, consider the same deign problem as in Example 1, but with the added requirement that the filtered sensor amplitude response be reasonably flat (i.e., without the noticeable higher frequency resonant rise in Fig. 3). To achieve the flat amplitude response we allow that the filtered sensor phase response be negative linear versus frequency (rather than zero) over the Example 1 specified design frequency range of 0 to 100 Hz. This is equivalent to introducing a fixed time delay in the filtered output signal for all frequency components within the 0 to 100 Hz band. (Note: In some digital filtering applications, a fixed time delay is acceptable, providing that the filter output tracks the sensor input in amplitude and relative phase. For example, in [2], the filtered accelerometer output is combined with an ideal gyro output to form a sculling correction.

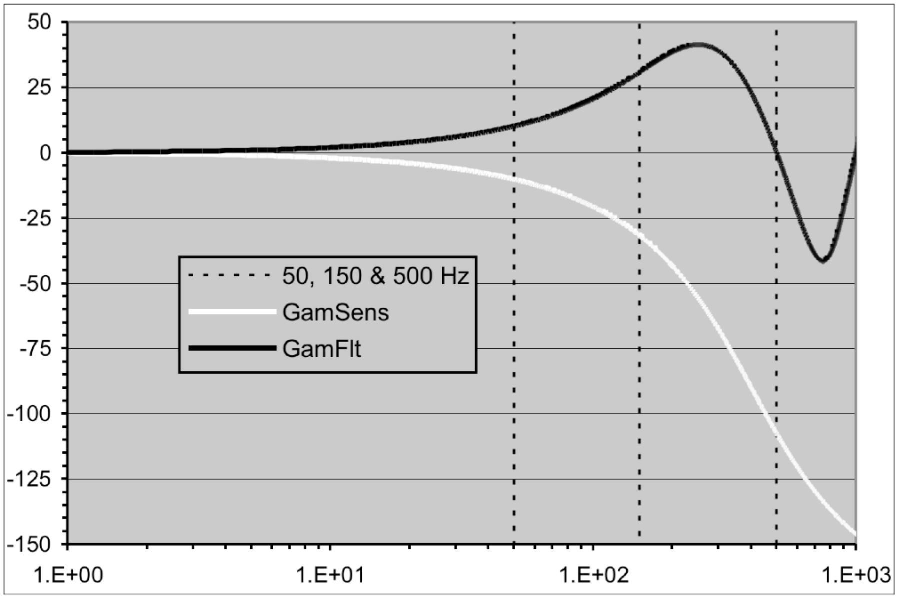

Fig. 1 - Sensor And Filter Phase Response Characteristics
$\gamma_{\mathrm{Sens}}(\Omega)$ (deg), $\gamma_{\mathrm{Flt}}(\Omega)$ (deg) vs. $2\pi\Omega$ (hz)

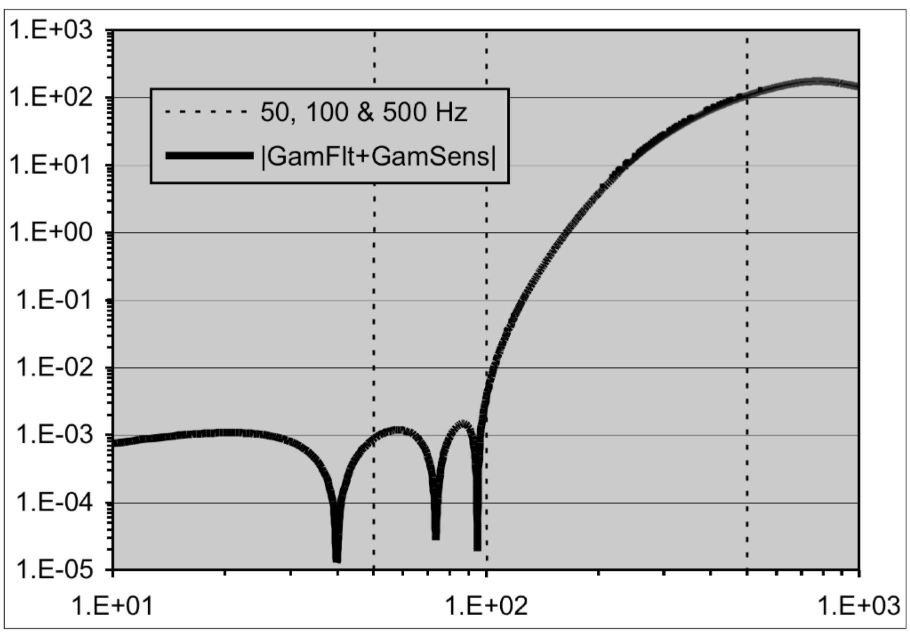

Fig. 2 - Filter Phase Compensation Effectiveness
$\left|\gamma_{\mathrm{Sens}}(\Omega)+\gamma_{\mathrm{Flt}}(\Omega)\right|$(deg) vs. $2\pi\Omega$ (hz)

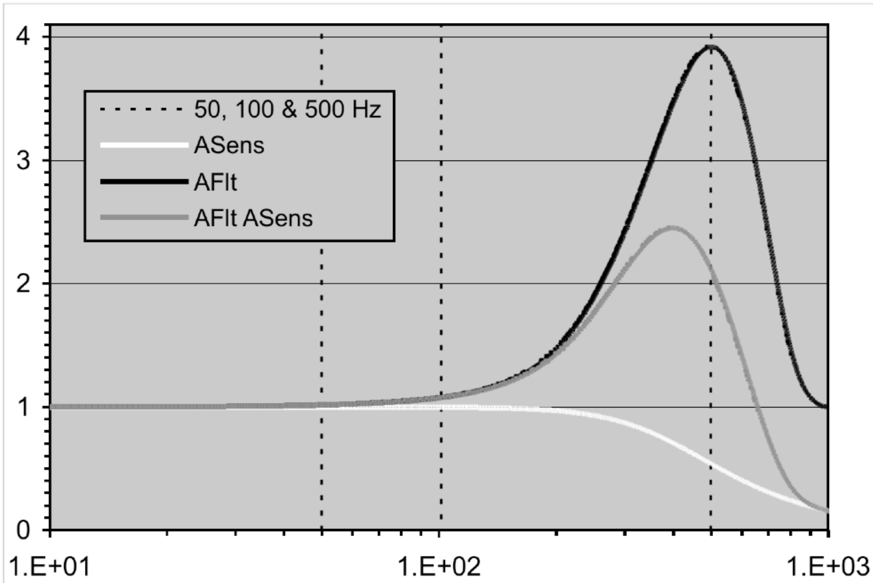

Fig. 3 - Corresponding Amplitude Responses
$A_{Sens}(\Omega)$ (dmls), $A_{Flt}(\Omega)$ (dmls), $A_{Flt}(\Omega)\, A_{Sens}(\Omega)$ (dmls) vs. $2\pi\Omega$ (hz)

If the accelerometer filtered output includes phase correction to a negative linear slope, the negative phase slope will not generate an error in the sculling calculation if the gyro input for sculling is time delayed by an amount corresponding to the filtered accelerometer negative linear phase slope.

Achieving the acceptable filter amplitude response is a numerical trial and error process whereby different negative linear phase slopes versus frequency are numerically tried. For the Example 1 problem, it was found that acceptable amplitude response performance could be obtained if $\gamma_{Desired}(\Omega)$ had a linear phase versus frequency Ω slope of $\gamma_{Desired}(\Omega) / \Omega = -0.75 \times [\pi / (2\, \omega_{Sens})]$. Figs. 4 - 6 show the resulting filtered sensor performance using this linear phase relationship. The associated (1) filter coefficients from (22) are as follows:

$$b_1 = 1.15934133 \quad b_2 = -0.235116786 \quad b_3 = 0.861694648 \quad b_4 = -0.0103940075 \qquad (25)$$

Fig. 4 shows the $A_{Sens}(\Omega)$ and $A_{Flt}(\Omega)$ amplitude responses from (23) and (6), and the resulting combined filtered sensor amplitude response $A_{Flt}(\Omega)\, A_{Sens}(\Omega)$ designed against the reasonably flat high frequency goal (compared with Fig. 3), using the (25) coefficients. Fig. 5 is a plot of $\gamma_{Sens}(\Omega)$ from (23) and $\gamma_{Flt}(\Omega)$ from (6) using the (25) coefficients, the previously described $\gamma_{Desired}(\Omega)$ requirement, and the resulting filter output phase $\left| \gamma_{Sens}(\Omega) + \gamma_{Flt}(\Omega) \right|$, all versus frequency (in Hz). Fig. 6 plots $\left| \gamma_{Sens}(\Omega) + \gamma_{Flt}(\Omega) - \gamma_{Desired}(\Omega) \right|$, the filter output error in matching the $\gamma_{Desired}(\Omega)$ linear phase response over the 0 - 100 Hz frequency range Ω_{Range}.

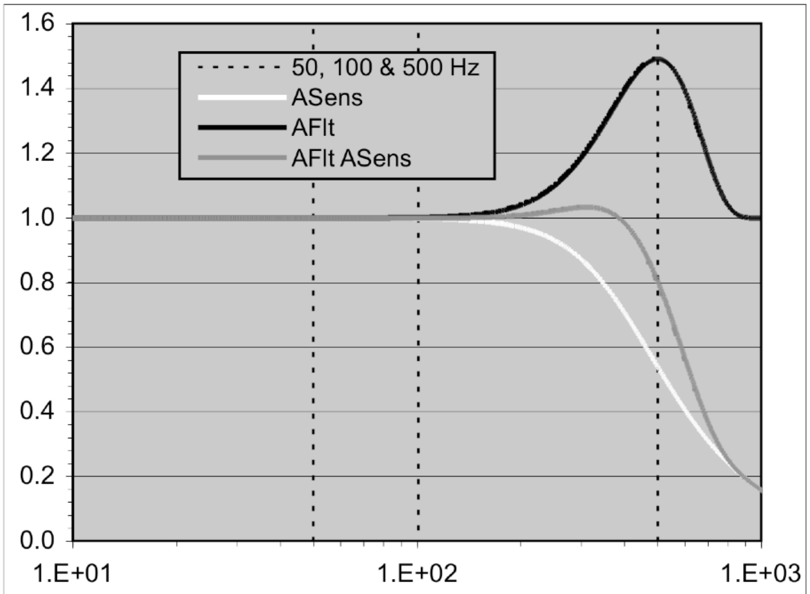

Fig. 4 - Sensor And Filter Amplitude Response
$A_{Sens}(\Omega)$ (dmls), $A_{Flt}(\Omega)$ (dmls), $A_{Flt}(\Omega)\,A_{Sens}(\Omega)$ (dmls) vs. $2\pi\Omega$ (hz)

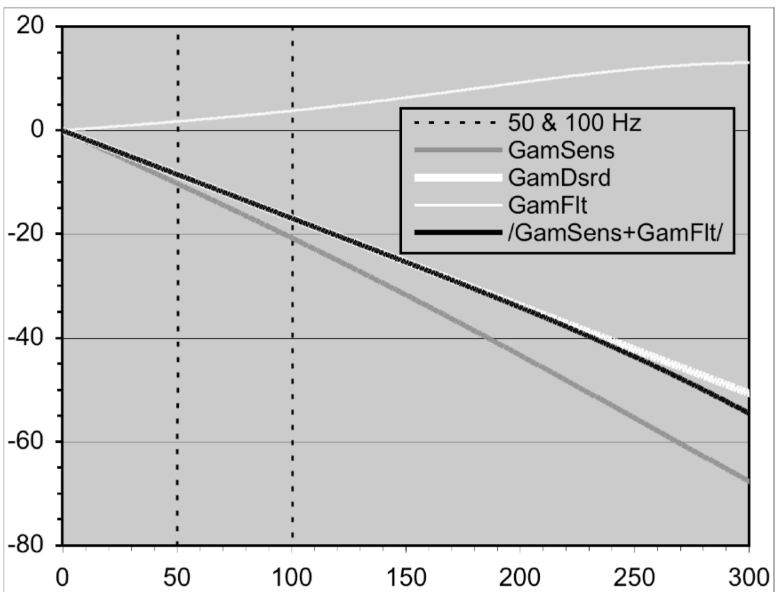

Fig. 5 - Sensor And Filter Phase Response
$\gamma_{Sens}(\Omega)$ (deg), $\gamma_{Flt}(\Omega)$ (deg), $\gamma_{Desired}(\Omega)$ (deg), $\left|\gamma_{Sens}(\Omega)+\gamma_{Flt}(\Omega)\right|$ (deg) vs. $2\pi\Omega$ (hz)

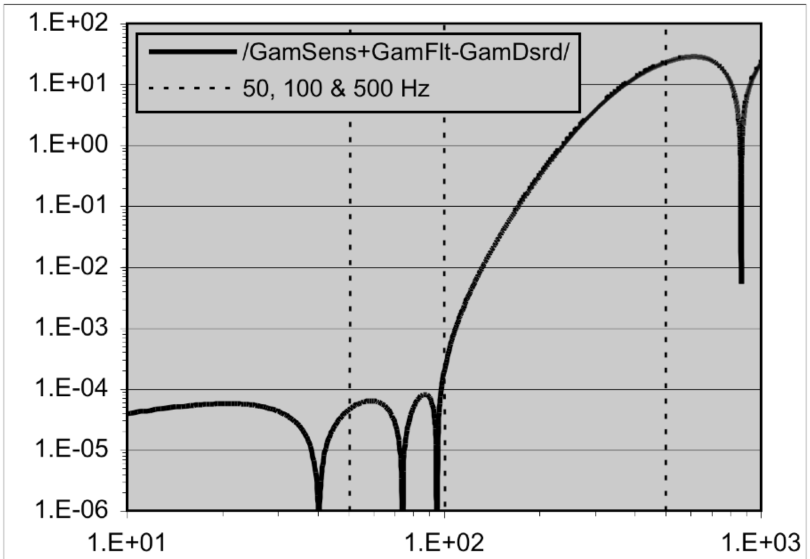

Fig. 6 - Filter Phase Compensation Effectiveness
$\left|\gamma_{Sens}(\Omega)+\gamma_{Flt}(\Omega)-\gamma_{Desired}(\Omega)\right|$ (deg) vs. $2\pi\Omega$ (hz)

CAUTIONARY NOTE

Numerical matrix inversion is required in this paper to compute filter coefficients. In some unusual cases, numerical round-off will impact results obtained (e.g., for large values of M which directly translates to the dimension of the matrix being inverted). A simple solution (as in [3]) is to maintain a small value of $W(\Omega)$ at high frequencies so that it never equals zero. It is recommended when computing coefficients that the matrix inversion routine be tested for accuracy by pre and post multiplying its output with the matrix being inverted to assure that the result (call it I*) equals identity within acceptable limits (e.g., so that each element of I* is within 1.E-5 of the correct identity matrix value).

CONCLUSIONS

Non-recursive fixed gain digital filter design can be based on achieving a specified dynamic phase response over a selected frequency range. This provides an alternative to past filter design approaches based on specified amplitude response or general dynamic characteristics. Some numerical experimentation may be required to achieve optimum results.

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